The Limited Power of Powering

Polynomial Identity Testing and a Depth-four Lower Bound for the Permanent

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Representation of Univariate Polynomials

$$P(X) = X^{10} + 5X^6 + 3X^2 + 1$$

Representations

- Dense: $[1, 0, 0, 0, 5, 0, 0, 0, 3, 0, 1]$
- Sparse: $\{(10, 1), (6, 5), (2, 3), (0, 1)\}$

Representation of Multivariate Polynomials

$$P(x, y, z) = x^5 y^3 z^2 + 5xy^4 z + 3yz + 1$$

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Sparse representation not always relevant either.
Supersparse (lacunary) representation and circuits.

Arithmetic Circuits

$$Q(x,y,z) = x^4 + 4\,x^3y + 6\,x^2y^2 + 4\,xy^3 + x^2z + 2\,xyz$$
$$+ y^2z + x^2 + y^4 + 2\,xy + y^2 + z^2 + 2\,z + 1$$

Arithmetic Circuits

$$Q(x, y, z) = (x + y)^4 + (z + 1)^2 + (x + y)^2(z + 1)$$

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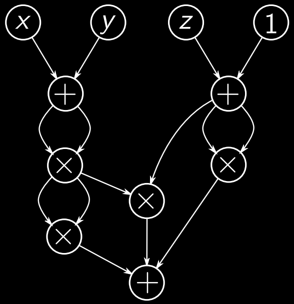

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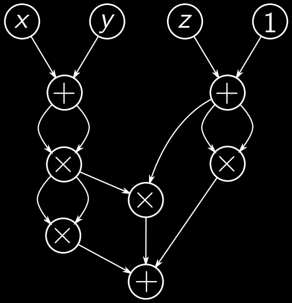

$\rightsquigarrow$ Straight Line Programs

Algebraic Complexity Theory

Complexity of a polynomial = size of its smallest circuit

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  - Polynomial complexity: Determinant

$$\det\left((x_{ij})_{1\leq i,j\leq n}\right) = \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}$$

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- Which polynomials have low/high complexity?
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Conjecture (Algebraic P $\neq$ NP)

The complexity of the permanent is super-polynomial.

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  - Roots finding, factorization, …

Polynomial Identity Testing

Theorem (Schwartz'80, Zippel'79, DeMillo-Lipton'78)

Let P be a non zero polynomial with n variables of total degree D, if $x_1, \ldots, x_n$ are randomly chosen in a set of integers S of size $\frac{D}{\epsilon}$ then the probability that $P(x_1, \ldots, x_n) = 0$ is bounded by ϵ.

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$\implies$ *Super-polynomial lower bound for the permanent*

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The τ-conjecture

Conjecture (Shub & Smale, 1995)

For any $f \in \mathbb{Z}[X]$ of complexity $\tau(f)$,

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Theorem (Bürgisser, 2006)

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Sum of products of sparse polynomials

Definition

Let $\mathrm{SPS}(k, m, t, A)$ the class of polynomials

$$f(X) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j(X)^{\alpha_{ij}}$$

where the $f_j \in \mathbb{R}[X]$ are t-sparse and $0 \leq \alpha_{ij} \leq A$.

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- Known techniques: $2^{\mathcal{O}((kmt)^2)}$ [Khovanskii'80, Risler'85]

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Conjecture (Koiran, 2011)

Let $f \in \mathrm{SPS}(k, m, t, A)$, then

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1. Upper bound on # real roots of $f \in \mathrm{SPS}(k, m, t, A)$
2. Lower bound for the permanent
3. Polynomial Identity Testing for SPS-like circuits

Upper bound for the number of real roots of SPS polynomials

Theorem

There exists $C > 0$ *such that the number of real roots of any* $f = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}} \in \mathrm{SPS}(k, m, t, A)$ *is at most*

$$C \cdot \left[e \cdot \left(1 + \frac{t^m}{2^{k-1} - 1} \right) \right]^{2^{k-1}-1}.$$

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- Independent of A.
- If k and m are fixed, this is polynomial in t.

Case $k = 2$

Proposition

The polynomial

$$f = \prod_{j=1}^{m} f_j^{\alpha_j} + \prod_{j=1}^{m} f_j^{\beta_j}$$

has at most $2mt^m + 4m(t-1)$ *real roots.*

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Proof sketch. Let $F = f / \prod_j f_j^{\alpha_j} = 1 + \prod_j f_j^{\beta_j - \alpha_j}$.

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$$F' = \underbrace{\prod_{j=1}^{m} f_j^{\beta_j - \alpha_j - 1}}_{\leq 2m(t-1) \text{ roots and poles}} \times \underbrace{\sum_{j=1}^{m} (\beta_j - \alpha_j) f_j' \prod_{l \neq j} f_l}_{\leq 2mt^m - 1 \text{ roots}} .$$

Case $k > 2$

- Reduce the number of terms from k to $k-1$:

$$f(X) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j(X)^{\alpha_{ij}}$$

$$f(X) / \prod_{j} f_j^{\alpha_{1j}} = 1 + \sum_{i=2}^{k} \prod_{j=1}^{m} f_j(X)^{\alpha_{ij} - \alpha_{1j}}$$

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  - Be clever : Pacal Koiran, Sébastien Tavenas and the wonderful Wronskian (coming next week).

The permanent family

$$\mathrm{PER}_n(x_{11},\ldots,x_{nn}) = \mathrm{per}\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^{n} x_{i\sigma(i)}$$

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$n \mapsto \tau(\mathrm{PER}_n)$ grows faster than any polynomial function.

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- The conjecture for depth-4 circuits implies the general case [Agrawal–Vinay'08, Koiran'11]

Multivariate SPS polynomials

Definition

$(P_n)_{n\geq 0} \in \mathrm{mSPS}(k, m)$ if

$$P_n(x_1, \ldots, x_{Q(n)}) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{j,n}^{\alpha_{ij,n}}(\vec{x})$$

where

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- exponential-size depth-4 circuits
- polynomial-size circuits with polynomial-depth

Lower bound for the permanent

Theorem

For any fixed k and m, (PER_n) *does not have* $\mathrm{mSPS}(k, m)$ *circuits.*

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Proof sketch. $(\mathrm{PER}_n) \in \mathrm{mSPS}(k,m)$

$$\implies \tau((\mathrm{PER}_n)) \leq \mathrm{poly}(n)$$

$$\implies \mathrm{PW}_n(X) = \prod_{i=1}^{2^n}(X-i) \in \mathrm{SPS}(k,m,\mathrm{poly}(n),2^{\mathrm{poly}(n)})$$

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But PW_n has 2^n roots: contradiction.

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Theorem

For fixed k and m, we can test for zero $f \in \mathsf{SPS}(k, m, t, A)$ in time polynomial in t and A.

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- Reduce the number of terms in the sum to 1.
- At each step, check if the monomial of larger degree vanishes.
- Compute the last term explicitely and check if it is zero.

A better PIT ?

Open Problem

What is the complexity to decide wether $\sum_{i=1}^{k} \prod_{j=1}^{m} a_{ij}^{\alpha_{ij}} \equiv 0$?

Special case of SPS$(k, m, 1, A)$ with only polynomials of degree 0.

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Proposition

With an oracle testing for zero $\sum_{i=1}^{k} \prod_{j=1}^{m} a_{ij}^{\alpha_{ij}}$, PIT algorithm in time polynomial in t and $\mathrm{bitsize}(A)$.

A better PIT ?

Open Problem

What is the complexity to decide wether $\sum_{i=1}^{k} \prod_{j=1}^{m} a_{ij}^{\alpha_{ij}} \equiv 0$?

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Remark. Works with mSPS polynomials by Kronecker substitution : $X_i \mapsto X^{(d+1)^i}$.

Conclusion

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