# Enumeration Complexity: Looking for Tractability

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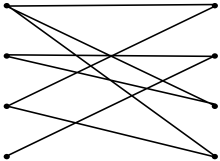
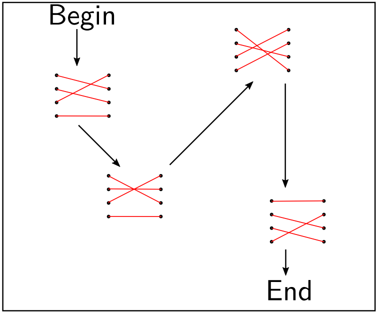
# Enumeration problems

- **Enumeration problems**: list all solutions rather than deciding whether there is one or finding one.
- **Complexity measures**: total time, delay between solutions, space.
- **Motivations**: database, logic, counting, optimization, biology, chemistry, datamining . . .

Input: a graph.



Output: perfect matchings.



# Framework

An enumeration problem $A$ is a function which associates to each input a set of solutions $A(x)$.

An enumeration algorithm must generate every element of $A(x)$ one after the other without repetition.

The computation model for enumeration is a RAM with uniform cost measure and an OUPTPUT instruction. Support efficient data structures.

**Complexity measures:**

- total time
- incremental time
- delay
- space

**Parameters:**

- input size
- output size
- single solution size

# Complexity classes

Several complexity classes introduced in the $80$'s [Johnson et al.] to answer the question **what is tractability** in enumeration?

1. Polynomially balanced predicate: EnumP
2. Output polynomial: OutputP
3. Incremental polynomial time: IncP
4. Polynomial delay: DelayP
5. Strong polynomial delay: SDelayP
6. Constant Delay: CD

# Polytime testing

**Definition**

A problem $A$ is in $\textsc{EnumP}$ if deciding whether $y \in A(x)$ is in P and if all $y \in A(x)$ are of polynomial size in $|x|$.

Equivalent of NP for enumeration.

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**Definition**

A **parsimonious reduction** from $A$ to $B$, two enumeration problems, is a pair of polynomial time computable functions $f, g$ such that for all $x$, $g(x)$ is a bijection from $B(f(x))$ to $A(x)$.

- Useful to prove hardness of enumerating solutions of NP-complete problems.
- Not general enough to prove hardness of natural problems.

# Tractability and EnumP

Restriction compared to the polynomial hierarchy for enumeration [Creignou et al.].

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Not a relevant notion of tractability:

1. No algorithm out of bruteforce.
2. Finding traces of SAT formulas or maximal H-free edge induced subgraphs are not in $\textsc{EnumP}$ but easy to solve.
3. Useful for hardness.

# Output polynomial

An output sensitive algorithm has its complexity depending on both its input and output.

**Definition**

A problem $A \in \textsc{EnumP}$ is in $\textsc{OutputP}$ if there is a polynomial $p$ and a machine $M$ which solves $A$ in total time $O(p(|x|, |A(x)|))$.

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$\textsc{OutputP} \neq \textsc{EnumP}$ iff P $\neq$ NP, using enumeration of solutions of any NP-complete problem.

# OutputP and tractability

Relevant measure of tractability because it depends on the number of solutions. Many limitations:

- All solutions must be computed (certificate of optimality, building a library).
- Should not be too many solutions and the degree of the polynomial complexity is critical.
- No hardness result and very few problems known in this class.

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**Question:** is there a natural problem in $\textsc{OutputP}$ but not in the classes below?

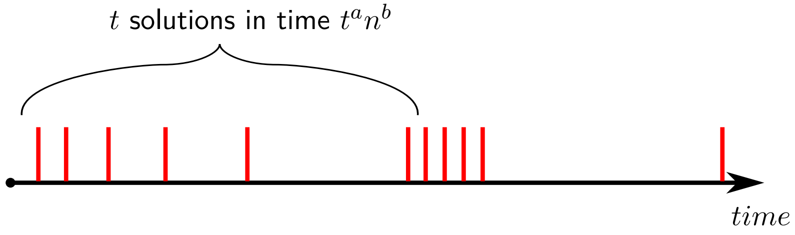
Dualization in distributive lattices [Defrain et al.].

# Incremental time

A machine $M$ enumerates $A$ in *incremental time* $f(t)g(n)$ if on every input $x$, $M$ enumerates $t$ elements of $A(x)$ in time $f(t)g(|x|)$ for every $t \leq |A(x)|$.

**Definition (Incremental polynomial time)**

$\textsc{IncP}$ is the set of enumeration problems such that there is an algorithm in incremental time $O(t^a n^b)$, for inputs of size $n$ and $a, b$ constants.



# Saturation algorithm

Many incremental polynomial time algorithms are saturation algorithms:

- **begin** with a polynomial number of simple solutions
- **for each** tuple of already generated solutions apply a rule to produce a new solution
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1. Accessible vertices in a graph by flooding.
2. Determinization of an automata.
3. Generating all the circuits of a matroid.
4. Generate all possible unions of sets.

# Relation to a search problem

Search problem $\textsc{AnotherSol}{\cdot}A$
**Input:** $x$ and a set of solutions $S \subset A(x)$
**Output:** $y \in A(x) \setminus S$ or $\sharp$ if there is none.

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**Theorem**

*An enumeration problem $A$ is in $\textsc{IncP}$ if and only if $\textsc{AnotherSol}{\cdot}A \in \mathsf{FP}$.*

**Hardness proofs**: maximal models of Horn formulas [Kavvadias et al.], dualization in distributive lattice [Babin and Kuznetsov, Defrain and Nourine], repairs in databases [Kimfield et al.].

# Relationship with total functions

**Definition**

A problem in TFNP is a polynomially balanced polynomial time predicate $A$ such that for all $x$, $A(x)$ is not empty. An algorithm solving $A$ must produce an element of $A(x)$ on input $x$.

$$\mathsf{TFNP} = \mathsf{FP}^{\mathsf{NP} \cap \mathsf{coNP}}$$

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**Proposition (Capelli, S. 2019)**

$\textsc{IncP} \neq \textsc{OutputP}$ *if and only if* TFNP $\neq$ FP.

**Proof:** $(\Rightarrow)$Remark that $\textsc{AnotherSol}{\cdot}A$ is a TFNP problem when $A \in \textsc{OutputP}$.
$(\Leftarrow)$ Use many distinct copies of $A(x)$ to obtain an $\textsc{OutputP}$ problem, an $\textsc{IncP}$ algorithm allows to find one solution in FP.

# Incremental Hierarchy

**Definition (Incremental polynomial time hierarchy)**

A problem $A \in \textsc{EnumP}$ is in $\textsc{IncP}_a$ if there is a machine $M$ which solves it in incremental time $O(t^a n^b)$ for some constant $b$.

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**Theorem (Capelli, S. 2019)**

*If $ETH$ holds, then $\textsc{IncP}_a \subsetneq \textsc{IncP}_b$ for all $a < b$.*

**Proof sketch:** Problem $Pad_t$, input $\varphi$ a CNF, with $2^{nt}$ trivial solutions and the models of $\varphi$ duplicated $2^n$ times.
Since $\textsc{IncP}_a = \textsc{IncP}_b$, $Pad_{b-1}$ gives a $O(2^{\frac{a}{b}n})$ algorithm to solve SAT.
Using the better SAT algorithm, we have $Pad_{\frac{a}{b^2}} \in \textsc{IncP}_b$. Repeat this trick to contradict $ETH$.

# Complete enumeration problem

**Corollary**

*If $ETH$ holds, then there is no problem complete for parsimonious reduction in* $\textsc{IncP}$.

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The result is true for most reductions (as soon as $\textsc{IncP}_a$ is stable under the reduction).

# IncP and Tractability

Relevant notion of tractability for several reasons:

- Partial enumeration: more time means more guaranteed solutions.
- Hardness results using $\textsc{AnotherSol}{\cdot}A$.
- A strict hierarchy to classify the complexity of problems inside $\textsc{IncP}$.
- The class $\textsc{IncP}_1$ as the *really tractable* problems: *linear incremental time* i.e. polynomial time per solution.

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Drawbacks:

- No complete problem for the class.
- Weak regularity of the enumeration process

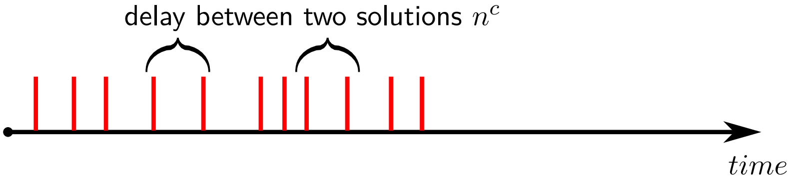
# Polynomial Delay

The delay is the maximum time between the production of two consecutive solutions given by an enumeration algorithm.

**Definition (Polynomial delay)**

A problem $A \in \textsc{EnumP}$ is in $\textsc{DelayP}$ if there is a machine $M$ which solves it on any input $x$ with delay $O(|x|^a)$.

$$\textsc{DelayP} \subseteq \textsc{IncP}_1$$



# Algorithmic Tricks for DelayP

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**Definition (Polynomial delay reduction)**

Reduction with only cartesian products and unions keeps $\textsc{DelayP}$ stable.

Similar to d-DNNF set circuits [Amarilli et al.].
Also in automata tools [Courcelle et al.], listing equivalence classes [Mary et al.].

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Trading space for regularity.

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Eliminating polynomial number of repetitions of solutions in a polynomial delay algorithm: exponential space.

**Cheater's lemma** [Carmeli et al.] and sampling to enumeration [Goldberg, Capelli and S.].

# DelayP and tractability

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Drawbacks:

- No method to prove hardness.
- Should restrict space to be relevant in practice.

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**Theorem (Capelli, S. 2019)**

*Let $A$ be a problem with a polynomial space incremental linear algorithm such that $\forall t < |A(x)|$, a polynomial fraction of the first $t$ solutions are generated with polynomial delay. Then $A \in \textsc{DelayP}^{poly}$.*

**Proof sketch:** Simulate the algorithm at different points in time and use the parts with high density of solutions to compensate for parts with low density.

# Regularization without space

**Theorem (Capelli, S. (unpublished))**

*An enumerator in incremental time $p(n)t$ and space $s(n)$ can be turned into an enumerator of delay $O(p(n) * log(N))$ and space $O(s(n) * \log(N))$, where $N$ is the number of produced solutions.*

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**Very Short Proof Sketch:**
Run $\log(N)$ copies of the enumerator. Each is in charge of the solutions in the interval of time $[2^i, 2^{i+1}]$. When a solution is found by one enumerator, it gives time to the enumerators in charge of larger intervals. Do not need to know $N$ nor $s(n)$ in advance.

# Complexity consequence

**Theorem (Capelli, S. (unpublished))**

$\textsc{DelayP}^{poly} = \textsc{IncP}_1^{poly}$

Three different takeaways:

- Incremental time is more relevant than delay.
- $\textsc{DelayP}$ is not relevant as a tractability notion: could be replaced by $\textsc{IncP}_1$.
- There is a good trick to help prove a problem is in $\textsc{DelayP}^{poly}$.

# Faster, better, tractabler

- $\textsc{DelayP}$ or $\textsc{IncP}_1$: the canonical notion of tractability for enumeration.
- $\textsc{SDelayP}$: polynomial delay in the size of the last solution.
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Help through relaxations:

- Randomized algorithms.
- Average delay: Total time / Number of solutions.
- Approximate enumeration.

# The class SDelayP

A precomputation time polynomial in the input is allowed.

**Definition (Strong polynomial delay)**

A problem $A \in \textsc{EnumP}$ is in $\textsc{SDelayP}$ if there is a machine $M$ which solves $A$ with delay $p(k)$, with $p$ a polynomial and $k$ the size of a solution.

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A few examples in $\textsc{SDelayP}$:

1. $s - t$ paths in a DAG
2. $MSO$ on graphs of bounded width [Courcelle]
3. $\exists FO +$ free second order variables [Durand, S.]
4. Saturation by set operations [Mary, S.]

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$\textsc{Enum}\cdot DNF$ is an interesting model to study the problem of non disjoint union:

- models of terms generated in constant delay and very structured
- interesting DNF subclasses
- $\textsc{Enum}\cdot DNF$ related to knowledge representation, minimal transversal enumeration, subset membership queries, CQ + SO variables, DNF model counting . . .

# Lower Bound Conjectures for SDelayP

Delay linear in $O(mn)$ by binary partition (similar to monotone CNF [Uno]).
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**DNF Enumeration Conjecture**

$\textsc{Enum}\cdot DNF \notin \textsc{SDelayP}$.

**Strong DNF Enumeration Conjecture**

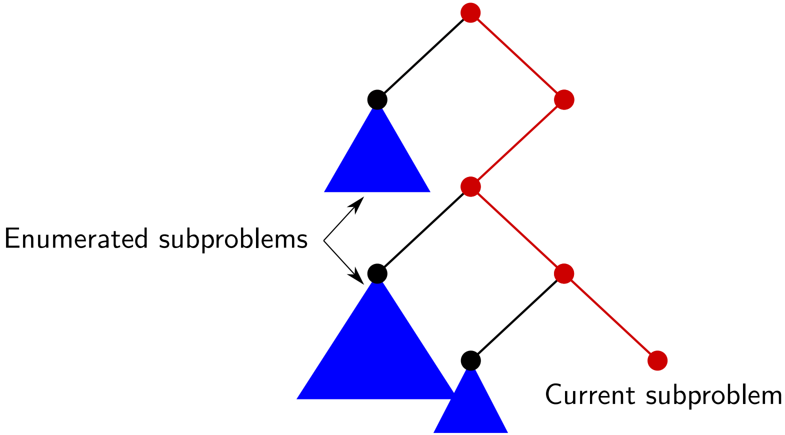
There is no algorithm generating the models of a DNF in delay $o(m)$ where $m$ is the number of terms.

# Results [Capelli, S. 2020]

| Class | Delay | Space |
|---|---|---|
| DNF | $O(\|\|D\|\|)$ | $O(\|\|D\|\|)$ |
| DNF | $O(nm^{1-\gamma})$ average delay | $O(\|\|D\|\|)$ |
| $k$-DNF | $k^{3/2}2^{2k}$ | $O(\|\|D\|\|)$ |
| Monotone DNF | $O(n^2)$, $m^2$ preprocessing | $O(sn)$ |
| Monotone DNF | $O(\log(mn))$ average delay | $O(mn)$ |

Table: Overview of the results. In this table, $D$ is a DNF, $n$ its number of variables, $m$ its number of terms and $s$ its number of models.
$\gamma = \log_3(2) > 0,63$

# Regularization of flashlight algorithms

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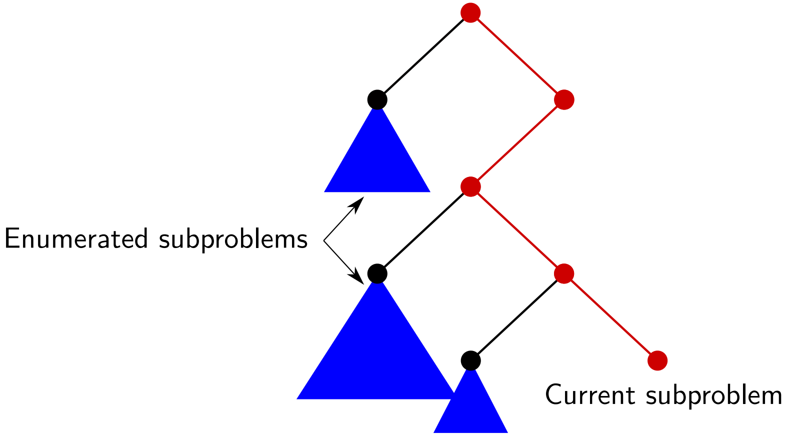


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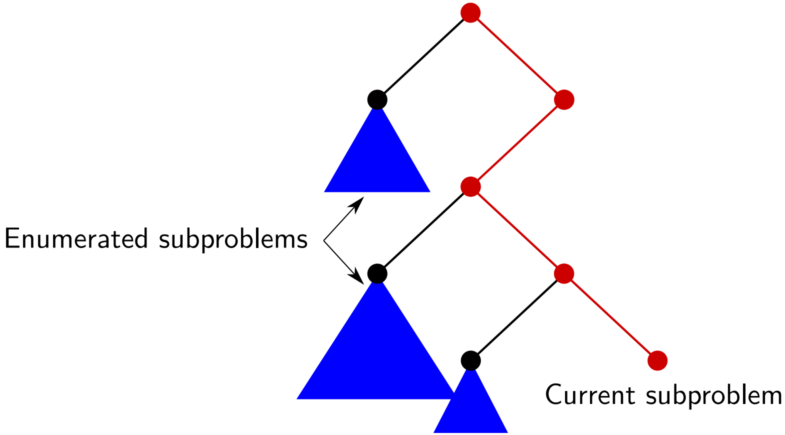


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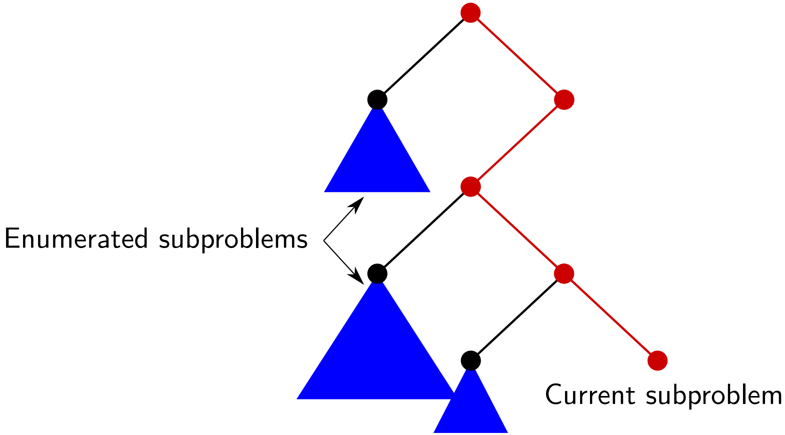


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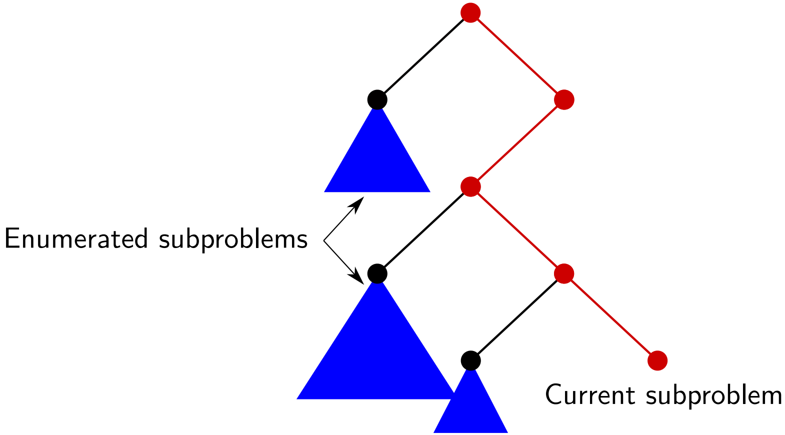


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- $\sum s_i$ solutions in $a \sum s_i + d$: $\textsc{IncP}_1$-enumerator.
- Regularized to a delay in $O(\log(N)a)$.

# Solving Enum$\cdot DNF$ with regularity

Applying the method of the previous slide to the algorithms designed for $\textsc{Enum}\cdot DNF$, we obtain the following theorems.

**Strong DNF Enumeration Conjecture is false**

There is an algorithm solving $\textsc{Enum}\cdot DNF$ in delay $O(n^2m^{1-\gamma})$ and linear space.

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**Theorem**

*There is an algorithm solving $\textsc{Enum}\cdot DNF$ for monotone formulas, in delay $\tilde{O}(n)$ and linear space.*

# SDelayP and tractability

It is a relevant notion of tractability when:

1. Large input with regard to the size of one solution: hypergraph problems, implicit input.
2. When solution size is "constant", could replace the "FPT" constant delay.
3. Doing infinite enumeration, the size of the solutions grows arbitrarily.
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4. Proving lower bound of the form $A \notin \mathrm{S}\textsc{Delay}\mathrm{P}$ should be easier.

Drawbacks:

1. In graph problems, the instance is typically of size $m = O(n^2)$ and the solutions are of size $n$: not a complexity problem.
2. Harder to obtain: not allowed to check the complete input between two solutions.
3. People are not familiar with this notion.

# Summary

$$\text{SDelayP} \subseteq \text{DelayP} = \text{IncP}_1 \subsetneq \text{IncP} \subsetneq \text{OutputP} \subsetneq \text{EnumP}$$

Conditional separation under complexity hypotheses: $\mathsf{P} \neq \mathsf{NP}$, $\mathsf{TFNP} \neq \mathsf{FP}$ and ETH.

# Summary

$$\textsc{SDelayP} \subsetneq \textsc{DelayP} \subsetneq \textsc{IncP} \subsetneq \textsc{OutputP}$$

If we remove the condition to be in $\textsc{EnumP}$: unconditional separation.

# Open problems: hardness

1. $\textsc{DelayP} \neq \textsc{SDelayP}$?
2. Existence of a complete problem in $\textsc{OutputP}$ or $\textsc{IncP}_1$?
3. Logical characterization of $\textsc{IncP}_1$, $\textsc{SDelayP}$?

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Lower bounds ($\textsc{SDelayP}$, $\textsc{IncP}_i$) or fine grained complexity for real problems:

1. Minimal hitting sets of hypergraphs: delay of $m^{O(\log(m))}$.
2. Minimal hitting sets of $k$-regular hypergraphs in $\textsc{IncP}_{k+2}$.
3. Maximal cliques of a graph in $\textsc{IncP}_1$.
4. Circuits of a binary matroids in $\textsc{IncP}_2$.
5. Models of a DNF in $\textsc{IncP}_1$.

Questions ???

# Four flavors of constant delay

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- Allow dynamic amortization (generalized OUPTUT instruction).
- Constant amortized time (CAT) algorithms. Generation of combinatorial structures of a given size, subgraphs of graphs. Pushout amortization [Uno].

# Four flavors of constant delay

The term constant delay is used to denote different things.

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- Allow dynamic amortization (generalized OUPTUT instruction).
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- FPT algorithm, arbitrary dependency in the parameter. Many examples from logic/database (data complexity)in surveys by [Segoufin, Durand]. Often polynomial number of solutions: restricting preprocessing is fundamental.