Complexity of enumeration: saturation problems

Yann Strozecki

Université de Versailles St-Quentin-en-Yvelines
Laboratoire DAVID

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Enumeration problems

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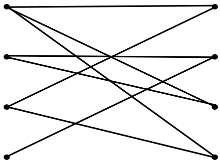
Perfect matching ?



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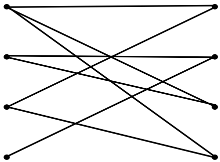

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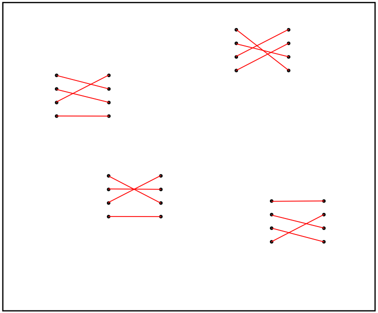

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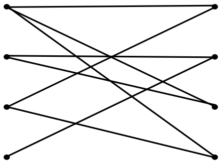

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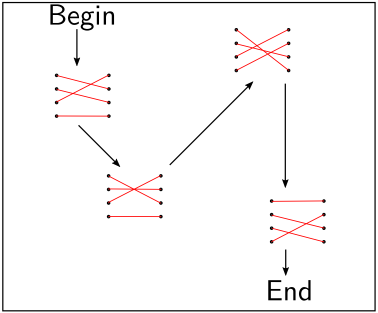

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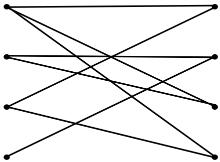

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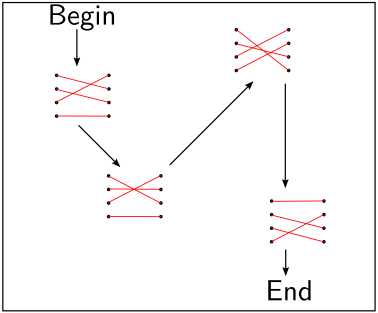

Enumeration problems

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- Complexity measures: total time and delay between solutions.
- Motivations: database queries, optimization, building libraries.

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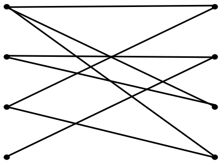

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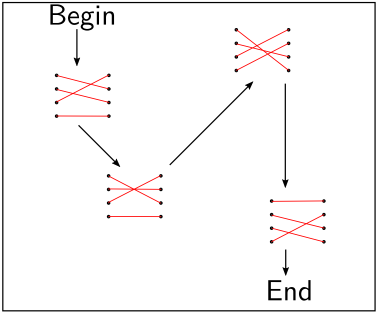

Framework

An enumeration problem A is a function which associates to each input a set of solutions $A(x)$.

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Concrete complexity classes:

A polynomial time precomputation is allowed.

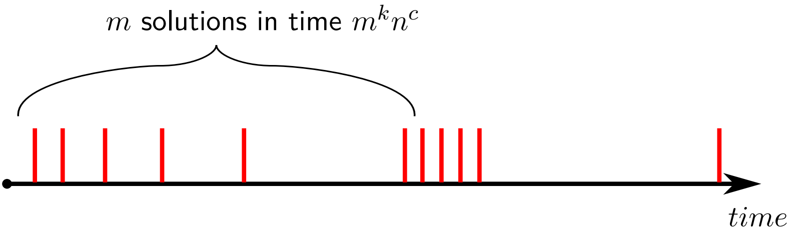
1. Polynomial total time: $\textsc{TotalP}$ (Minimal hitting set)
2. Incremental polynomial time: $\textsc{IncP}$
3. Polynomial delay: $\textsc{DelayP}$
4. Constant delay with a precomputation step: $\textsc{Constant-Delay}_{poly}$ (Database queries)

Incremental time

Definition (Incremental polynomial time)

$\mathbf{IncP_k}$ is the set of enumeration problems such that there is an algorithm which for all m produces m solutions (if they exist) from an input of size n in time $O(m^k n^c)$ with c a constant.

$$\textsc{IncP} = \bigcup_{k \geq 1} \mathbf{IncP_k}$$



Relation to research problem

Definition

The decision problem $\textsc{AnotherSol}\cdot A$ is given an instance x and a set of solutions S in $A(x)$, find a solution not in S if there is one.

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An enumeration problem A is in $\textsc{IncP}$ *if and only if* $\textsc{AnotherSol}{\cdot}A$ *can be solved in polynomial time.*

The other enumeration classes cannot be related to decision problems. Hard to use classical notions such as completeness.

Saturation algorithm

Most algorithms with an incremental delay are saturation algorithms:

- **begin** with a polynomial number of simple solutions
- **for each** k-uple of already generated solutions apply a rule to produce a new solution
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   - $\{12, 134, 23, 14\}$
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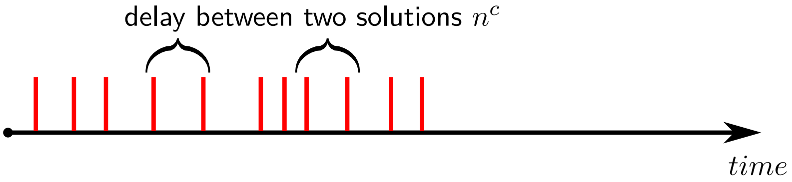
Polynomial Delay

The delay is the maximum time between the production of two consecutive solutions in an enumeration.

Definition (Polynomial delay)

$\textsc{DelayP}$ is the set of enumeration problems such that there is an algorithm whose delay is polynomial in the input.

$$\textsc{DelayP} \subseteq \textsc{IncP}_1$$



Unions in polynomial delay

Closure by union revisited.

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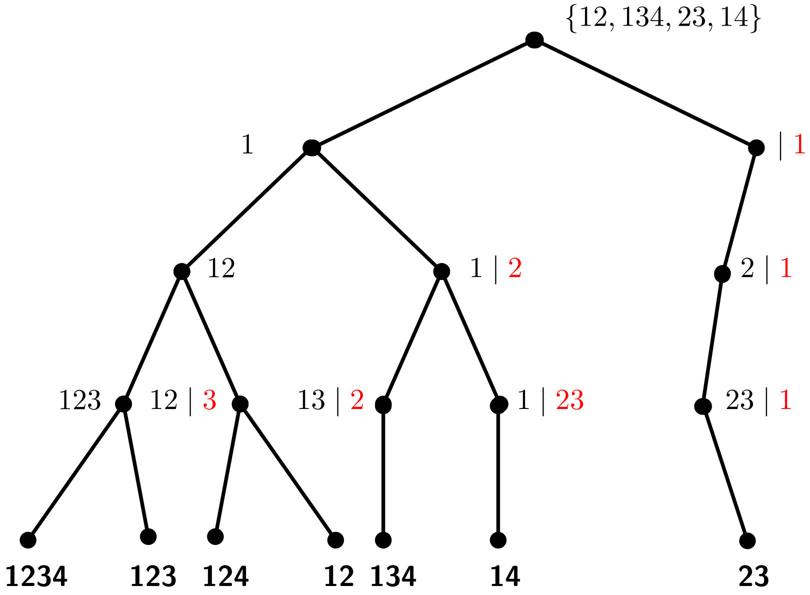
4. This problem is easy to solve in time $O(mn)$.

Partial solution tree



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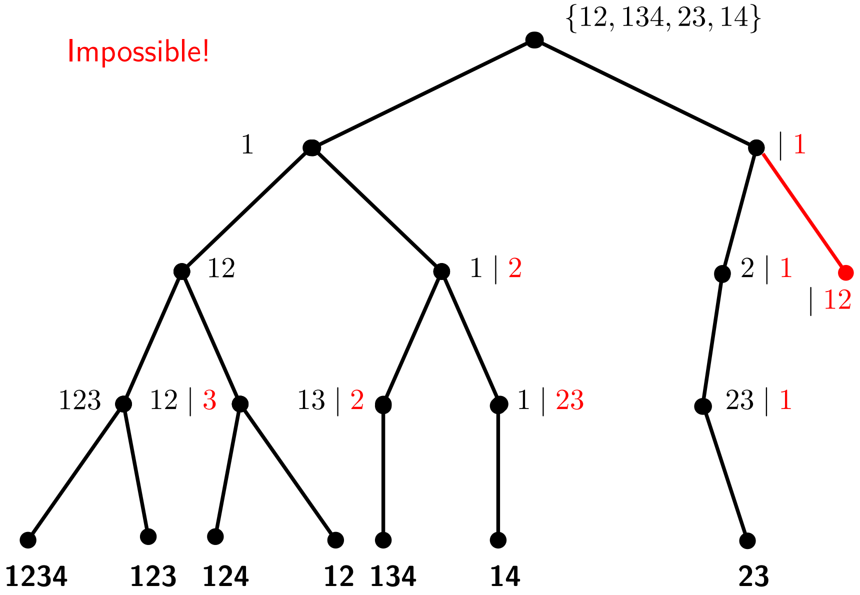

Backtrack search

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Many applications:

- Generate all subgraphs with some constraints.
- Interpolate polynomials.
- Fold graphs.
- Generate solutions of formulas.

Can be improved by playing with the order of the variable chosen to be fixed.

Separation of complexity classes

Separation:

$$\textsc{DelayP} \subsetneq \textsc{IncP} \subsetneq \textsc{TotalP} \subsetneq \textsc{EnumP}$$

Conditional separation under complexity hypotheses.

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Conditional separation under complexity hypotheses.

1. If $\mathsf{P} = \mathsf{NP}$ everything collapses.
2. $\text{IncP} \neq \text{TotalP}$ if $\mathsf{P} \neq \mathsf{NP} \cap \mathsf{coNP}$ using problems with always a solution but an hard to find one.
3. $\text{TotalP} \neq \text{EnumP}$ if $\mathsf{P} \neq \mathsf{NP}$, using enumeration of solutions of any NP-complete problem.

Separation modulo Exponential time hypothesis

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The **E**xponential **T**ime **H**ypothesis states that 3-SAT has no algorithm in time $2^{o(n)}$ where n is the number of variables.

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Theorem (Capelli, Durand, S.)

ETH implies $\textsc{IncP}_i \subsetneq \textsc{IncP}_{i+1}$ *for all* i *and thus* $\textsc{IncP} \neq \textsc{DelayP}$.

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$\textsc{DelayP} = \textsc{IncP}_1$ using an exponential size balanced binary search tree.

Open problem: is it true in polynomial space ?

From saturation to polynomial delay

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To make the question interesting and tractable we need to restrict the saturation rules. Since it works for the union, we will consider set operations.
Our aim is to design the largest toolbox of efficient enumeration algorithms.

Set operations

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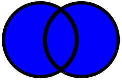
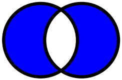
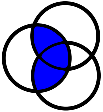
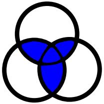
$\vee$	$\begin{pmatrix}1\\0\\1\end{pmatrix} \vee \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$	$\cup$	
$+$	$\begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$	$\triangle$	
$x \wedge (y \vee z)$	$\begin{pmatrix}1\\0\\1\end{pmatrix} \wedge (\begin{pmatrix}1\\0\\0\end{pmatrix} \vee \begin{pmatrix}1\\1\\0\end{pmatrix}) = \begin{pmatrix}1\\0\\0\end{pmatrix}$	???	
$\mathrm{maj}(x, y, z)$	$\mathrm{maj}(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}) = \begin{pmatrix}1\\0\\0\end{pmatrix}$	Majority	

Closure by set operation

Let $\mathcal{S}$ be a set of boolean vectors of size n and $\mathcal{F}$ be a finite set of boolean operations.

Closure:

- $\mathcal{F}^0(\mathcal{S}) = \mathcal{S}$
- $\mathcal{F}^i(\mathcal{S}) = \{f(v_1, \ldots, v_t) \mid v_1, \ldots, v_t \in \mathcal{F}^{i-1}(S)$ and $f \in \mathcal{F}\}$
- $Cl_{\mathcal{F}}(\mathcal{S}) = \cup_i \mathcal{F}^i(\mathcal{S})$

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Our enumeration problem is then to compute $Cl_{\mathcal{F}}(\mathcal{S})$. It can be seen as computing:

- the closure of a boolean relation by polymorphisms,
- the closure of a set system by set operations,
- the smallest hypergraph with some properties which extends the input hypergraph.

Extension problem

$\textsc{Closure}_{\mathcal{F}}$:

Input: $\mathcal{S}$ a set of vectors of size n, and a vector v of size n
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Goal: prove that **Closure**$_{\mathcal{F}} \in \mathsf{P}$ for as many sets $\mathcal{F}$ as possible, to use the backtrack search.

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For instance $(x \vee y) + x + z \ \in < \vee, + >$.

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Lemma

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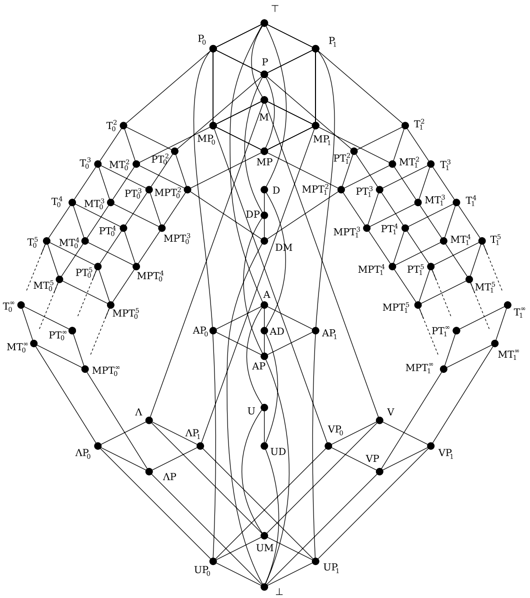
There are less clones than families and they are well described and organized in Post's lattice.

Post's lattice



How to reduce Post's lattice

To an operation f we can associate its dual $\overline{f}$ defined by $\overline{f}(s_1, \ldots, s_t) = \neg f(\neg s_1, \ldots, \neg s_t)$.

Proposition

The following problems can be polynomially reduced to $\textsc{Closure}_{\mathcal{F}}$*:*

1. $\textsc{Closure}_{\mathcal{F}\cup\{\mathbf{0}\}}$, $\textsc{Closure}_{\mathcal{F}\cup\{\mathbf{1}\}}$, $\textsc{Closure}_{\mathcal{F}\cup\{\mathbf{0},\mathbf{1}\}}$
2. $\textsc{Closure}_{\overline{\mathcal{F}}}$
3. $\textsc{Closure}_{\mathcal{F}\cup\{\neg\}}$ *when* $\mathcal{F} = \overline{\mathcal{F}}$

Reduced Post's lattice

Clone	Base
I_2	$\emptyset$
L_2	$x+y+z$
L_0	$+$
E_2	$\wedge$
S_{10}	$x \wedge (y \vee z)$
S_{10}^k	$Th_k^{k+1}, x \wedge (y \vee z)$
S_{12}	$x \wedge (y \rightarrow z)$
S_{12}^k	$Th_k^{k+1}, x \wedge (y \rightarrow z)$
D_2	maj
D_1	maj$, x+y+z$
M_2	$\vee, \wedge$
R_2	$x\,?\,y\;:\;z$
R_0	$\vee, +$

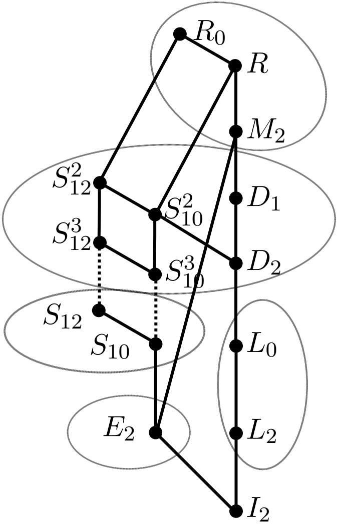


Figure: Reduced Post's lattice, the edges represent inclusions of clones

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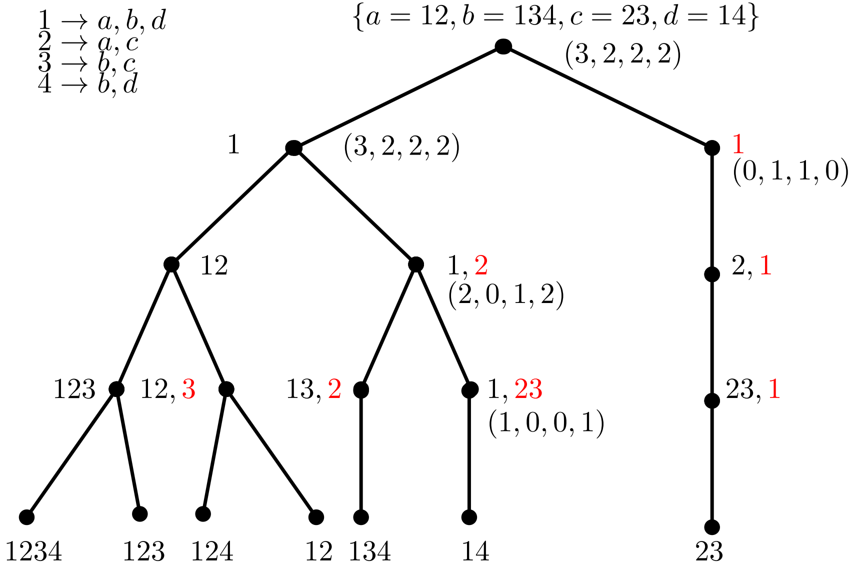
- Complexity comes from solving repeatedly the extension problem.
- We can set up datastructures to solve it faster.
- During a branch of the backtrack search we go over the instance once.
- Therefore the delay is improved to $O(mn)$.

Open question: can we get rid or decrease the dependency on m?

The data structures



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4. Same idea for $L_2 =< x + y + z >$, with an additionnal constraint on the elements of the basis.

All boolean functions

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- The problem $\textsc{Closure}_{M_2,\neg}$ is in P.

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- **Instance:** a set of boolean vectors $\mathcal{S} = \{s_1, \ldots, s_n\}$.
- We can use the distributivity of $\vee$ and $\wedge$ to get the following normal form of a solution: $\vee_i \wedge_j l_{ij}$ with l_{ij} an element of S or its negation.
- The problem $\textsc{Closure}_{M_2,\neg}$ is in P.
- We can compute the minimal $\wedge_j l_{ij}$, they are the atoms of the boolean algebra $\textsc{Closure}_{M_2,\neg}(\mathcal{S})$, which can be generated with delay $O(n)$ by Gray code.

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- This can be done for M_2, $R_2 =< x\,?\,y : z >$ and $R_0 =< \vee, + >$.

Majority

Proposition

Let $\mathcal{S}$ be a vector set, a vector v belongs to $Cl_{<maj>}(\mathcal{S})$ if and only if for all $i, j \in [n]$, $i \neq j$, there exists $x \in \mathcal{S}$ such that $x_{i,j} = v_{i,j}$.

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- A linear number of pairs must be checked when a single coefficient is fixed, delay $O(mn^2)$.
- For each pair of indices, we can precompute the possible pairs of values, delay $O(n^2)$.

Thank universal algebra

Definition

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Theorem (Baker-Pixley)

Let $\mathcal{F}$ be a clone which contains a near unanimity term of arity k, then $v \in Cl_{\mathcal{F}}(\mathcal{S})$ if and only if for all set of indices I of size $k-1$, $v_I \in Cl_{\mathcal{F}}(\mathcal{S})_I = Cl_{\mathcal{F}}(\mathcal{S}_I)$.

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The *threshold* function of arity k, denoted by Th^k_{k-1} is equal to 1 if and only if at least $k-1$ of its k arguments are equal to 1.

Corollary

For all clones $\mathcal{F}$ containing Th^k_{k-1}, $\textsc{Closure}_{\mathcal{F}} \in$ P

The result

There are two special cases $S_{10} =< x \wedge (y \vee z) >$ and $S_{12} =< x \wedge (y \to z) >$ whose proofs are similar to but not implied by the previous case.

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Theorem (Mary,S.)

For all sets $\mathcal{F}$ of boolean operations, $\textsc{Closure}_{\mathcal{F}} \in \mathsf{P}$.

Corollary

For all sets $\mathcal{F}$ of boolean operations, enumerating $Cl_{\mathcal{F}}$ is in $\textsc{DelayP}$.

Larger domains

The natural generalization is to consider vectors over a finite domain D with more than two elements. Every definition generalizes in a straigthforward manner.

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The operations are now from D^k to D and there are many more of them. In particular the lattice of clones is uncountable and we cannot do a case by case proof.

- $D = \{0, 1, 2\}$
- $f(x, y) = x + y$ when $x + y <= 2$
- $f(x, y) = 2$
- $\textsc{Closure}_{<f>}$ is NP-hard

Tractable cases

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2. If the operation is a commutative group operation, the closure problem is in polynomial time. It can be reduced to solving several linear systems.

3. Associative operations yields polynomial delay algorithms by using the reverse search. It is just a depth first traversal of the solutions which can be organized as a graph of low degree. However the memory used is exponential.

Take away

Results:

- $\textsc{Closure}_{\mathcal{F}} \in$ P for all sets $\mathcal{F}$ of boolean operations.
- Enumeration of $Cl_{\mathcal{F}}$ with delay $O(n^a)$ except when $\mathcal{F} =< \vee >$.
- $\textsc{Closure}_{\mathcal{F}}$ can be NP-hard for three elements domain.

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- $\textsc{Closure}_{\mathcal{F}} \in \mathsf{P}$ for all sets $\mathcal{F}$ of boolean operations.
- Enumeration of $Cl_{\mathcal{F}}$ with delay $O(n^a)$ except when $\mathcal{F} =< \vee >$.
- $\textsc{Closure}_{\mathcal{F}}$ can be NP-hard for three elements domain.

Open questions:

- Characterize the complexity of $\textsc{Closure}_{\mathcal{F}}$ for larger domains (dichotomy theorem?).
- Enumerate $Cl_{\mathcal{F}}$ when $\mathcal{F}$ is a single non commutative group operation.
- Improve the delay of enumerating $Cl_{<\vee>}$.

Thanks !

Questions ?