Monadic second-order model-checking on decomposable matroids

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Brno

Matroids have been design to abstract the notion of dependence.

Definition

A matroid is a pair $(E, \mathcal{I})$, E is a finite set and $\mathcal{I}$ is included in the power set of E. Elements of $\mathcal{I}$ are said to be independent sets, the others are dependent sets.

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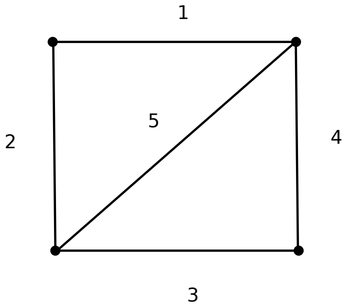

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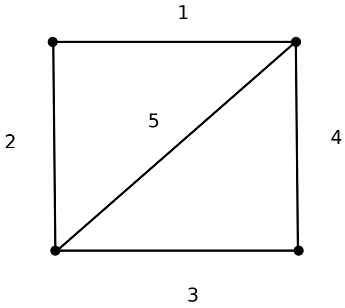

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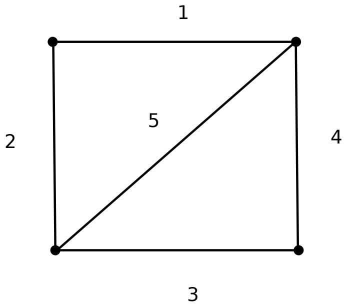

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Let A be a matrix, the ground set of the matroid defined on A is the set of the column vectors and a set of column vectors is independent if they are linearly independent. It is called a *vector matroid*.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

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The monadic second-order logic (MSO_M) on matroids is defined from the following relations :

1. $=$, the equality for element and set of the matroid
2. $e \in F$, where e is an element of the set F
3. $indep(F)$, where F is a set and the predicate is true iff F is an independent set of the matroid

Being a circuit :

$$\neg indep(X) \wedge \forall Y \, (Y \not\subseteq X \vee X = Y \vee indep(Y))$$

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Introduction to Matroids

Matroid Decomposition

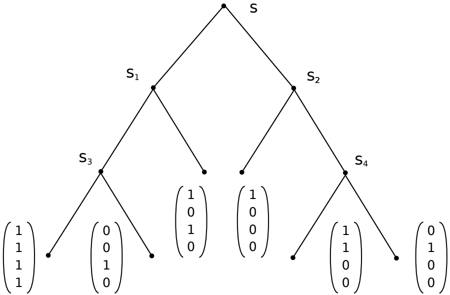
From MSO_M to MSO on enhanced trees

Matroid grammars

Definition

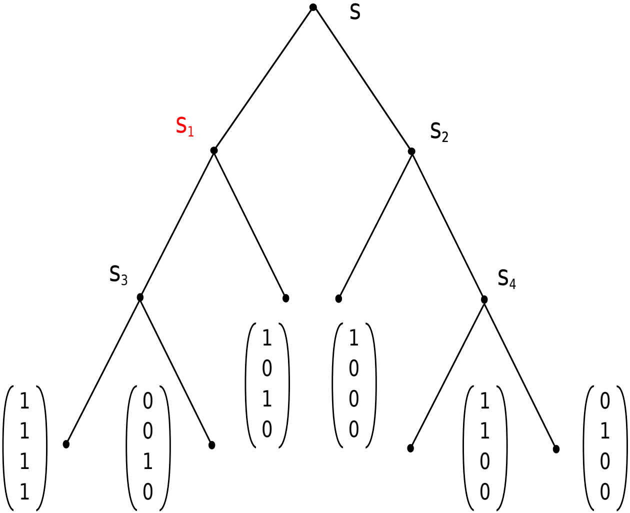
A branch decomposition of a matroid represented by the matrix X is a tree whose leaves are in bijection with the columns of X.

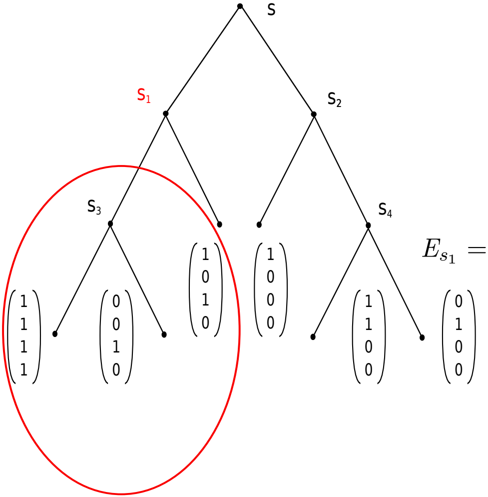
$$X = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$



Three important spaces are defined at each node s of the tree :

- E_s is the subspace generated by all the leaves of the tree rooted in s
- E_s^c is the subspace generated by all the leaves not in the tree rooted in s


S
S1
S2
S3
S4
(1 1 1 1)
(0 0 1 0)
(1 0 1 0)
(1 0 0 0)
(1 1 0 0)
(0 1 0 0)



$$E_{s_1} = < \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} >$$

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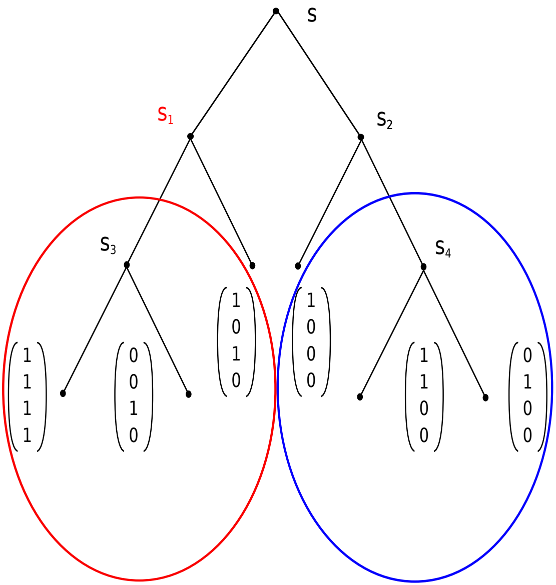

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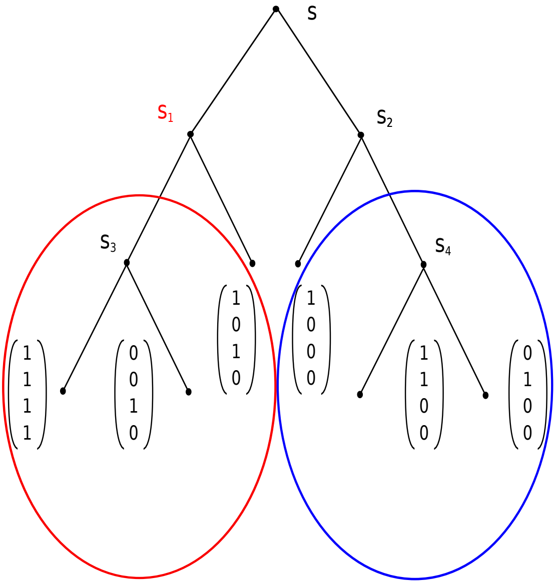

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Theorem (Hliněný and Oum)

Let $\mathbb{F}$ be a finite field, t a constant and M a $\mathbb{F}$-matroid of size n. There is an algorithm in time $O(n^3)$ which gives a branch decomposition tree of width at most $3t$ if the branch-width of M is at most $t+1$. If the branch-width is more than $t+1$, the algorithm may stop with no output.

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Representing the matroid by local information : Enhanced Tree

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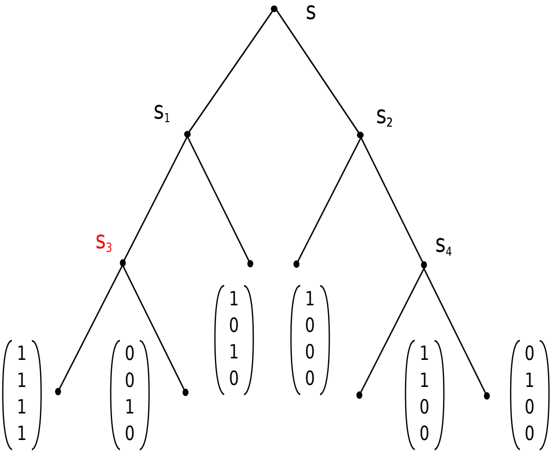

$$B_{s_3} = < \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} >$$

$$C_{s_3} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

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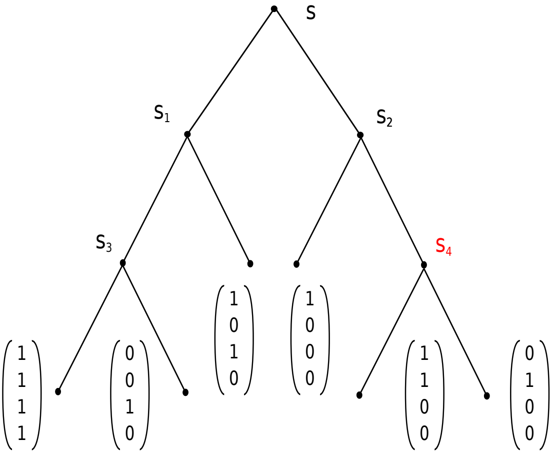

$$B_{s_4} = < \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} >$$

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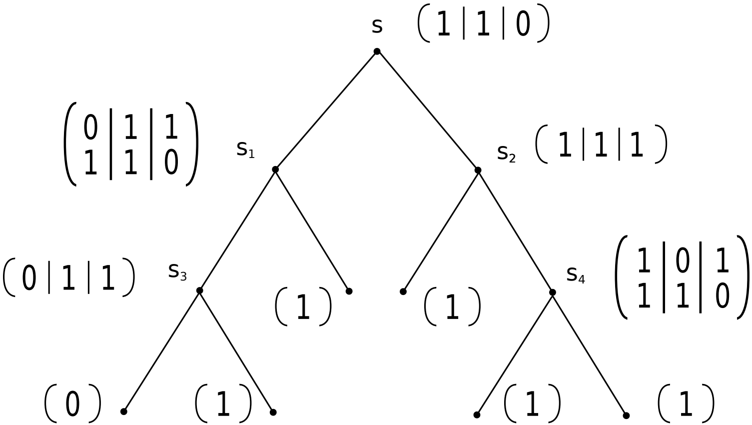

s
(1 | 1 | 0)
s_1
(0 | 1 | 1 / 1 | 1 | 0)
s_2
(1 | 1 | 1)
s_3
(0 | 1 | 1)
(1)
(1)
s_4
(1 | 0 | 1 / 1 | 1 | 0)
(0)
(1)
(1)
(1)

We now want to prove the following theorem :

Theorem (Hliněný 2006)

The model checking problem for MSO_M is decidable in time $f(t,k,l) \times n^3$ over the set of representable matroids, where f is a computable function, k the size of the field, t the branch-width and l the size of the formula.

We now want to prove the following theorem :

Theorem

Let M be a matroid of branch-width less than t, T one of its enhanced tree and $\phi(\vec{x})$ a MSO_M formula with free variables $\vec{x}$, we have

$$(M, \vec{a}) \models \phi(\vec{x}) \Leftrightarrow (T, f(\vec{a})) \models F(\phi(\vec{x}))$$

Aim : a characterization of the dependent sets.

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Definition (Signature of a set)

Let A be a matrix representing a matroid and T one of its enhanced tree. Let s be a node of T and let X be a subset of the leaves of T_s which are seen as columns of A. Let v be an element of B_s, obtained by a nontrivial linear combination of elements of X. Let $c_1, \ldots, c_l$ denote the column vectors of the third part of C_s. They form a base of B_s. Thus there is a signature $\lambda = (\lambda_1, \ldots, \lambda_l)$ such that $v = \sum_{i=1}^{l} \lambda_i c_i$. We say that X admits the *signature* λ at s.

The set X also always admits the signature $\varnothing$ at s.

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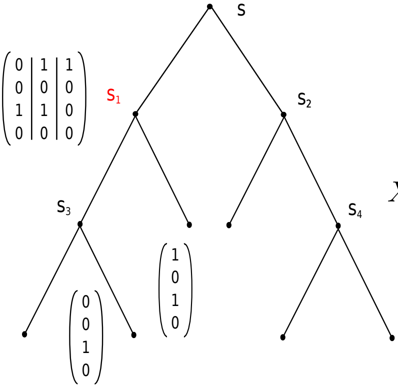
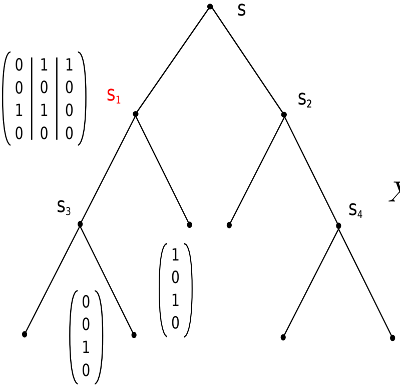


$$X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



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Definition

Let N be a matrix over $\mathbb{F}$ divided in three parts $(N_1|N_2|N_3)$, and let λ, μ, δ be three signatures over $\mathbb{F}$. The submatrix N_i has l_i columns, and its j^{th} vector is denoted by N_i^j. The relation $R(N, \lambda, \mu, \delta)$ is true if :

- $\lambda = \mu = \delta = \varnothing$ or
- λ and at least one of μ, δ are not $\varnothing$ and the following equation holds

$$\sum_{i=1}^{l_1} \mu_i N_1^i + \sum_{j=1}^{l_2} \delta_j N_2^j = \sum_{k=1}^{l_3} \lambda_k N_3^k \tag{1}$$

If a signature is $\varnothing$, the corresponding sum in Eq. 1 is replaced by 0.

Local characterization :

Lemma

Let T be an enhanced tree, s one of its nodes with children s_1, s_2 and N the label of s. Let X_1 and X_2 be two sets of leaves chosen amongst the leaves of T_{s_1} and T_{s_2} respectively. If X_1 admits μ at s_1, X_2 admits δ at s_2 and $R(N, \lambda, \mu, \delta)$ holds then $X = X_1 \cup X_2$ admits λ at s.

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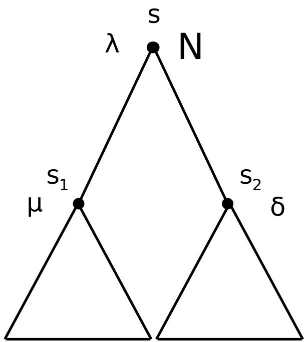
s
λ
N
s₁
μ
s₂
δ

Theorem (Characterization of dependency)

Let A be a matrix representing a matroid M, T one of its enhanced tree and X a set of columns of A. The set X is dependent if and only if there exists a signature λ_s for each node s of the tree T such that :

1. *for every node s labeled by N with children s_1 and s_2, $R(N, \lambda_s, \lambda_{s_1}, \lambda_{s_2})$ holds.*
2. *for every leaf s, $\lambda_s \neq \varnothing$ only if s is in bijection with an element of X and s is labeled by the matrix (α) with $\alpha \neq 0$.*
3. *the signature at the root is $(0, \ldots, 0)$*

- These signatures are represented by the set $\vec{X}$ of set variables X_λ indexed by all signatures λ of size at most t.
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- Consistency :

$$\Omega(\vec{X_\lambda}) = \forall s \bigvee_{\lambda} \left(X_\lambda(s) \bigwedge_{\lambda' \neq \lambda} \neg X_{\lambda'}(s) \right)$$

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First condition, for every node s labeled by N with children s_1 and s_2, $R(N, \lambda_s, \lambda_{s_1}, \lambda_{s_2})$ holds :

$$\Psi_1(\vec{X}) \equiv \forall s \, \neg leaf(s) \Rightarrow [\exists s_1, s_2 \, lchild(s, s_1) \wedge rchild(s, s_2)$$

$$\bigwedge_{\lambda_1,\lambda_2,\lambda,N} (label(s) = N \wedge X_{\lambda_1}(s_1) \wedge X_{\lambda_2}(s_2) \wedge X_{\lambda}(s)) \Rightarrow R(N, \lambda, \lambda_1, \lambda_2)]$$

Second condition, for every leaf s, $\lambda_s \neq \varnothing$ only if s is in bijection with an element of X and s is labeled by the matrix (α) with $\alpha \neq 0$:

$$\Psi_2(Y, \vec{X}) \equiv \forall s \, (leaf(s) \wedge \neg X_\varnothing(s)) \Rightarrow (\, Y(s) \wedge label(s) \neq (0))$$

Third condition, the signature at the root is $(0, \ldots, 0)$:

$$\Psi_3(\vec{X}) \equiv \exists s \, root(s) \wedge X_{(0,\ldots,0)}(s)$$

By combination of the three previous formulas we obtain a MSO formula for $Indep(X)$, of size $O((k+1)^{9t^2+3t})$.

We build the MSO formula $F(\phi)$ by induction : relativization to the leaves and the predicate indep is replaced by the formula $Indep$.

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We have then proved

Theorem

Let M be a matroid of branch-width less than t, $\bar{T}$ one of its enhanced tree and $\phi(\vec{x})$ a MSO_M formula with free variables $\vec{x}$, we have

$$(M, \vec{a}) \models \phi(\vec{x}) \Leftrightarrow (\bar{T}, f(\vec{a})) \models F(\phi(\vec{x}))$$

An application :

Theorem (Courcelle)

Let $\phi(X_1, \ldots, X_n)$ be a MSO formula with free variables. For every tree t, there exists a linear delay enumeration algorithm of the $X_1, \ldots, X_n$ such that $t \models \phi(X_1, \ldots, X_n)$ with preprocessing time $\mathcal{O}(|t| \times ht(t))$.

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Let $\phi(X_1, \ldots, X_n)$ be an MSO_M formula, for every matroid of branch-width t, the enumeration of the sets satisfying ϕ can be done with linear delay after a cubic preprocessing time.

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The same theorems can be proved for matroids equipped with unary predicates (colored matroids).

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Input : a matroid M and a set A of its elements
Output : accept if there is a circuit C of M such that $A \subseteq C$

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Generalisation of very natural problems and decidable in linear time over matroids of bounded branch-width. Interesting enumeration version.

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Input : a matroid M and a set A of its elements
Output : accept if there is a circuit C of M such that $A \subseteq C$

Generalisation of very natural problems and decidable in linear time over matroids of bounded branch-width. Interesting enumeration version.

Introduction to Matroids

Matroid Decomposition

From MSO_M to MSO on enhanced trees

Matroid grammars

Build a set of matroids which are not representable.

Definition (Boundaried matroid)

A pair (M, γ) is called a t boundaried matroid if M is a matroid and γ is an injective function from $[1, t]$ to M whose image is an independent set. The elements of the image of γ are called boundary elements and the others are called internal elements.

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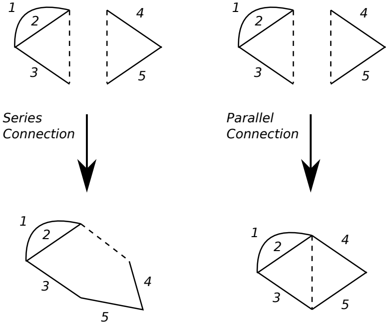

Definition

Let M_1 and M_2 be two 1 boundaried matroids of ground set S_1 and S_2. Their respective boundaries are $\{p_1\}$ and $\{p_2\}$. We denote by $\mathcal{C}(M)$ the collection of circuits of the matroid M. Let E be the set $S_1 \cup S_2 \cup \{p\} \setminus \{p_1, p_2\}$. We define two collections of subsets of E :

$$C_S = \begin{cases} \mathcal{C}(M_1 \setminus \{p_1\}) \cup \mathcal{C}(M_2 \setminus \{p_2\}) \\ \cup \{C_1 \setminus \{p_1\} \cup C_2 \setminus \{p_2\} \cup \{p\} \mid p_i \in C_i \in \mathcal{C}(M_i)\} \end{cases}$$

$$C_P = \begin{cases} \mathcal{C}(M_1 \setminus \{p_1\}) \cup \mathcal{C}(M_2 \setminus \{p_2\}) \\ \cup_{i=1,2} \ \{C_i \setminus \{p_i\} \cup \{p\} \mid p_i \in C_i \in \mathcal{C}(M_i)\} \\ \cup \{C_1 \setminus \{p_1\} \cup C_2 \setminus \{p_2\} \mid p_i \in C_i \in \mathcal{C}(M_i)\} \end{cases}$$

Definition

We write $M_1 \oplus_p M_2$ for the parallel connection of M_1 and M_2 restricted to the ground set $S_1 \cup S_2 \setminus \{p_1, p_2\}$ (one removes the boundary $\{p\}$).

Definition

Let M be a matroid and let γ_i^M for $i = 1, 2, 3$ be three injective functions from $[1, t_i]$ to the ground set of M. If the sets $\gamma_i^M([1, t_i])$ are independent and form a partition of the columns of M, then $(M, \{\gamma_i^M\}_{i=1,2,3})$ is called a 3-*partitioned matroid*.

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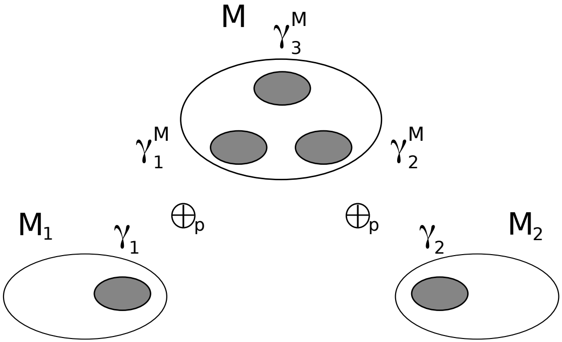
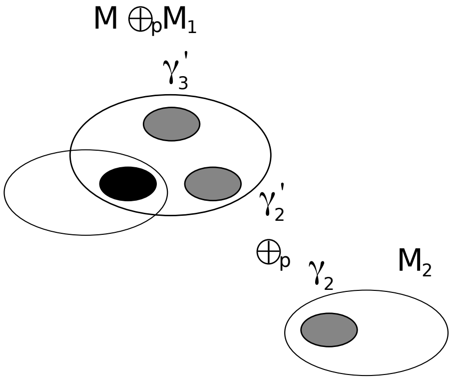
Definition

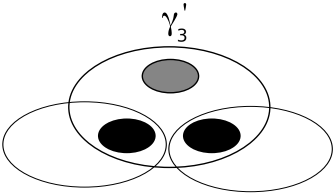
Let $\overline{M_1} = (M_1, \gamma_1)$ and $\overline{M_2} = (M_2, \gamma_2)$ be respectively a t_1 and a t_2 boundaried matroid and let M be a 3-partitioned matroid. We call $\overline{M_1} \odot_M \overline{M_2}$ the t_3 boundaried matroid defined by $(\overline{M_1} \oplus_p (M, \gamma_1^M), \gamma_2^M) \oplus_p \overline{M_2}$ with boundary γ_3^M.


M
γ_3^M
γ_1^M
γ_2^M
$\oplus_p$
$\oplus_p$
M_1
γ_1
γ_2
M_2



$M \oplus_p M_1$
γ_3'
γ_2'
$\oplus_p$
γ_2
M_2

$$M_1 \odot_M M_2$$



Definition

Let $\mathcal{L}_k$ be the set of 1 boundaried matroids of size at most k and let $\mathcal{M}$ be the set of 3-partitioned matroids of size 3. We write $\mathcal{T}_k$ for the set of terms $T(\mathcal{L}_k, \mathcal{M})$.

Aim, prove the following theorem :

Theorem

Let T be a term of $\tilde{\mathcal{T}}_k$ which represents the matroid M. Let f be the bijection between the leaves of T and the elements of M then $M \models \phi(\vec{a}) \Leftrightarrow T \models F(\phi(f(\vec{a})))$.

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Definition (Signature)

Let T be a term whose value is a boundaried matroid M and let X be a set of internal elements of M. The signature of the set X in T is the set of all the subsets S of the boundary such that $X \cup S$ is a dependent set in M.

1. if X is dependent then it is of signature $\{\{\}, \{1\}\}$ that we denote by $\mathbf{2}$
2. if X is dependent only when we add the boundary element then it is of signature $\{\{1\}\}$ which we denote by $\mathbf{1}$
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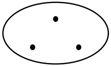
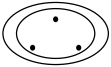
N_1



$$R_p(\cdot,\cdot,\mathbf{2},N_1) = \{(\mathbf{0},\mathbf{2}),(\mathbf{2},\mathbf{0}),(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{2})\}$$
$$R_p(\cdot,\cdot,\mathbf{1},N_1) = \{\}$$

N_2



$$R_p(\cdot,\cdot,\mathbf{2},N_2) = \{(\mathbf{0},\mathbf{2}),(\mathbf{2},\mathbf{0}),(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{2})\}$$
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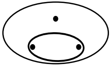
N_3



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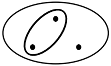
N_4



$$R_p(\cdot,\cdot,\mathbf{2},N_4) = \{(\mathbf{0},\mathbf{2}),(\mathbf{2},\mathbf{0}),(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{2})\}$$
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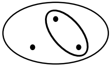
N_5



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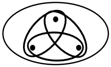
N_6



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Theorem (Characterization of dependency)

Let T be a term of $\tilde{\mathcal{T}}_k$ which represents the matroid M and let X be a set of elements of M. The set X is dependent if and only if there exists a signature λ_s at each node s of T seen as a labeled tree :

1. *if s_1 and s_2 are the children of s of label $\odot_N$ then $R_p(\lambda_{s_1}, \lambda_{s_2}, \lambda_s, N)$*
2. *if s is labeled by an abstract boundaried matroid N, then $X \cap N$ is a set of signature λ_s in N*
3. *the signature at the root is $\mathbf{2}$*

To prove the theorem, two variations :

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Corollary

The model-checking problem of MSO_M is decidable in time $f(k,l) \times n$ over the set of matroids given by a term of $\mathcal{T}_k$, where n is the number of elements in the matroid, l is the size of the formula and f is a computable function.

We can also give an operation on matrices to characterize the matroids of bounded branch-width.

Use this formalism to study broader classes or different classes (between the cycle matroids and the vector matroids).

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Thanks for listening !