Simple stochastic games: a state of the art

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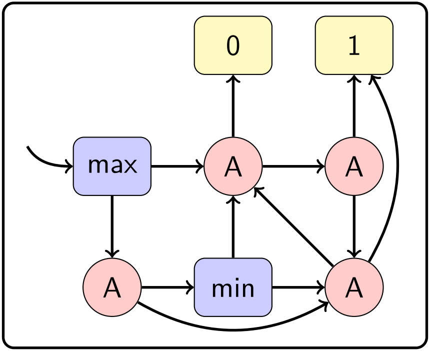
Simple stochastic game (SSG)

A Simple Stochastic Game (Shapley, Condon) is defined by a directed graph with:

- three sets of vertices V_{MAX}, V_{MIN}, V_{AVE} of outdegree 2
- two 'sink' vertices 0 and 1



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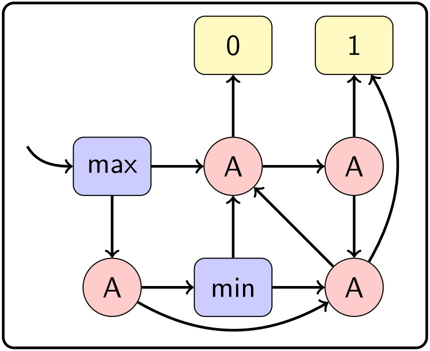

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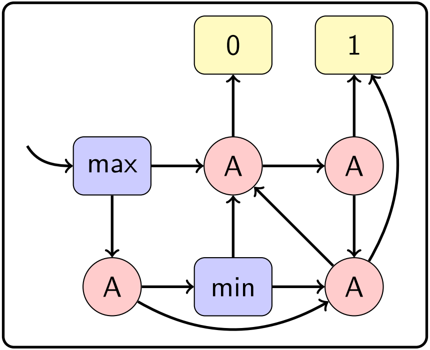
Semantic of SSGs

A play consists in moving a *pebble* on the graph:

- player MAX wants to reach the 1 sink
- player MIN wants to prevent him from doing so



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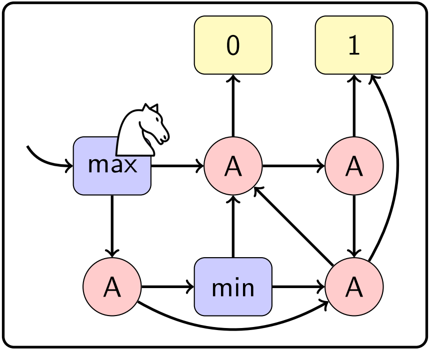

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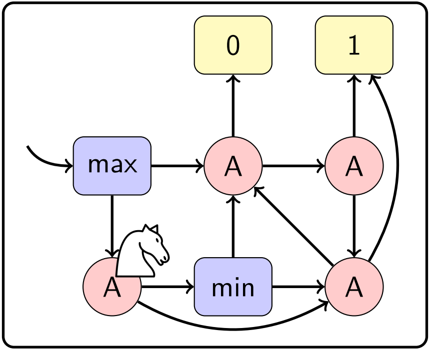

On a AVE node the next vertex is randomly determined.

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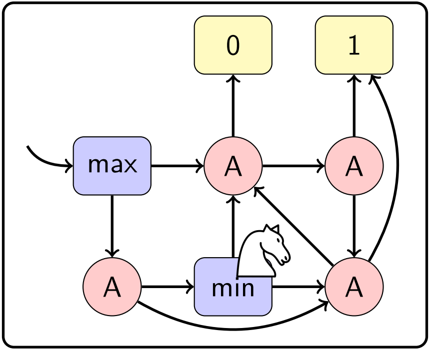

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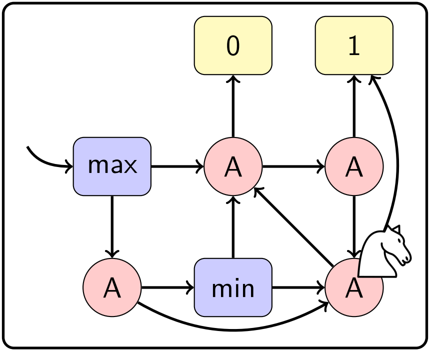

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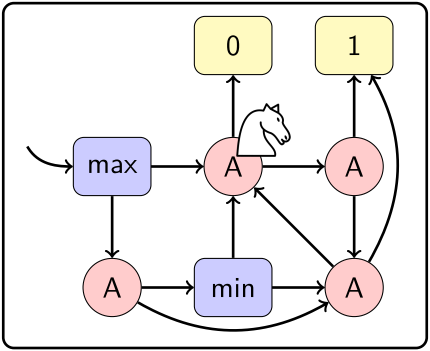

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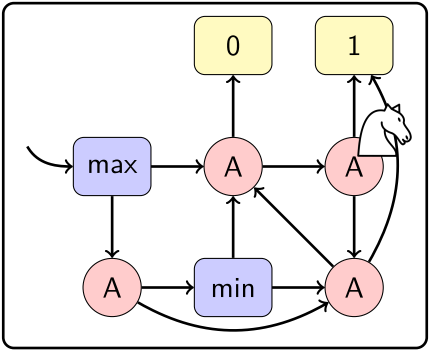

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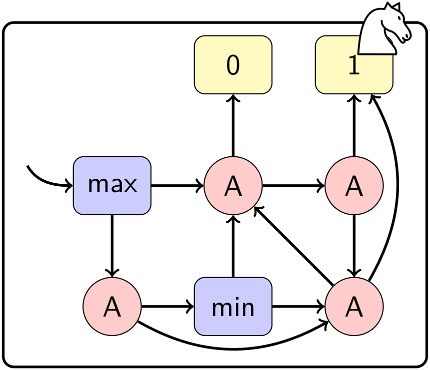

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Strategies and values

General definition of a strategy σ for a player MAX:

$$\sigma : \text{partial play ending in } V_{MAX} \longmapsto \text{probability distribution on outneighbours}$$

The value of a vertex x is:

$$v(x) = \sup_{\substack{\sigma \text{ strategy} \\ \text{for } \mathrm{MAX}}} \underbrace{\inf_{\substack{\tau \text{ strategy} \\ \text{for } \mathrm{MIN}}} \underbrace{\mathbb{P}_{\sigma,\tau}\left(1 \text{ is reached} \;\middle|\; \text{game starts in } x\right)}_{v_{\sigma,\tau}(x)}}_{v_\sigma(x)}$$

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Discounted payoff games

Played on a graph as SSGs, but no sink vertices nor average vertices.

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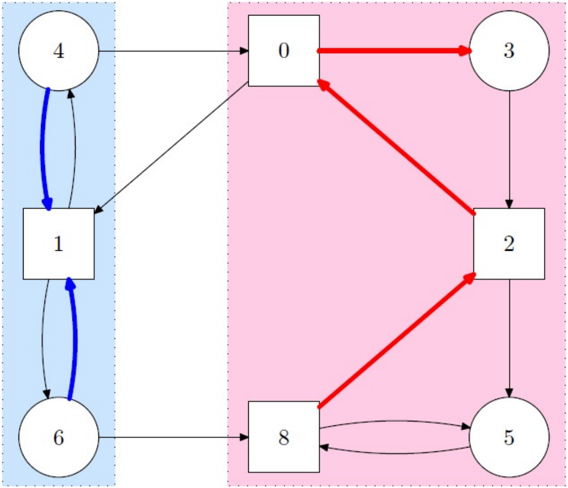
Parity games



- Two player game on a graph
- Play goes on forever
- Every vertex has a priority
- P set of infinitely seen priority

If the largest value of P is even, player 0 wins otherwise 1 wins.

Reduction from Parity games to SSGs

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Idea:

- Add two sinks 0 and 1.
- Assign for every transition a small probability to go to sink 0 (nodes of player 0) or sink 1 (nodes of player 1).
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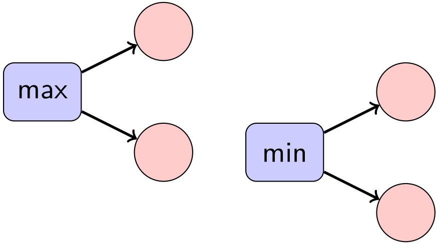
Simpler strategies

To compute values we can restrict our strategies to be

- *pure*: deterministic
- *memoryless*: does not depend from the memory

We call them positional strategies for short.

$$\sigma : V_{MAX} \longrightarrow V, \quad \tau : V_{MIN} \longrightarrow V$$



Forgetting the past makes it simpler

Pure:
Let σ be a randomized strategy which on vertex a chooses with probability λ the vertex b and with probability $1 - \lambda$ the vertex c.

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Minimax Theorem

Theorem (Condon 89)

For all vertices x,

$$\begin{aligned} v(x) &= \max_{\substack{\sigma \text{ positional strategy} \\ \text{for MAX}}} \; \min_{\substack{\tau \text{ positional strategy} \\ \text{for MIN}}} \; v_{\sigma,\tau}(x) \\ &= \min_{\substack{\tau \text{ positional strategy} \\ \text{for MIN}}} \; \max_{\substack{\sigma \text{ positional strategy} \\ \text{for MAX}}} \; v_{\sigma,\tau}(x) \end{aligned}$$

Main lines of a proof ...

1. Sups and infs are maxs and mins: optimal strategies and best responses exists (compacity and continuity arguments)
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Stopping SSGs

A SSG is stopping if for all strategies, the game reaches a sink vertex almost surely.

Theorem (Condon 89)

For every SSG G, there is a polynomial-time computable SSG G' such that

- *G' is stopping*
- *size of G' = poly(size of G)*
- *for all vertices x, $v_{G'}(x) > \frac{1}{2}$ if and only if $v_G(x) > \frac{1}{2}$*

How to stop a game?

Idea of proof:

1. $v_G(x) > \frac{1}{2} \iff v_G(x) \geq \frac{1}{2} + 4^{-n}$
2. values are stable under perturbations,

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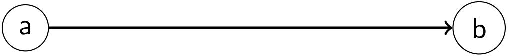

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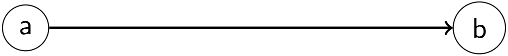

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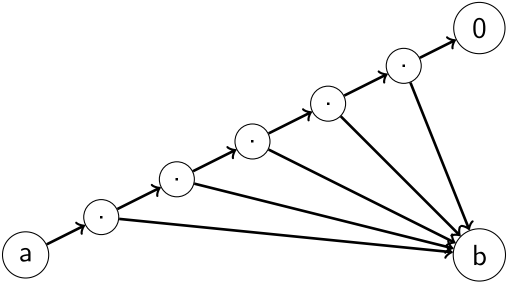

Optimality conditions

A language L is in NP if there is a language $C \in$ P such that

$$x \in L \Leftrightarrow \exists y \in \Sigma^{poly(|x|)}, (x, y) \in C$$

Lemma

G stopping SSG, and σ, τ are optimal strategies if and only if

$$\text{for all } x \in V_{MIN}, \quad v_{\sigma,\tau}(x) = \min(v_{\sigma,\tau}(x_1), v_{\sigma,\tau}(x_2))$$

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Introduction to Games

Fundamental Properties of SSGs and Complexity Classes

Algorithms to solve SSG

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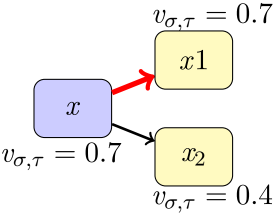

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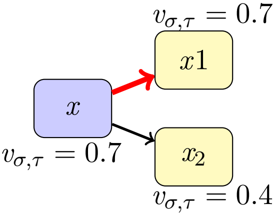

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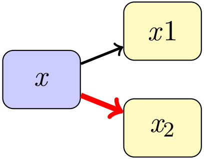

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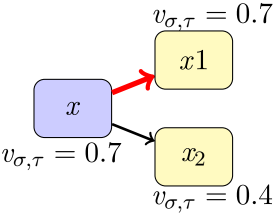

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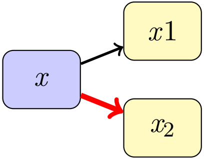

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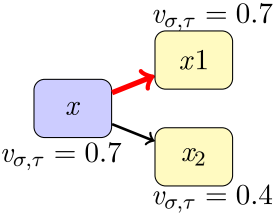

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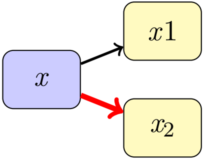

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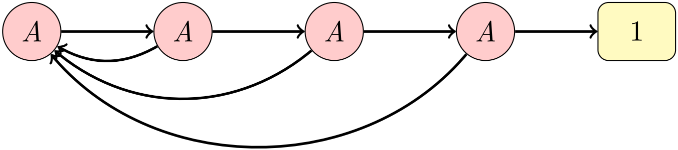

Hoffman-Karp Algorithm

The strategy improvement algorithm or Hoffman-Karp algorithm:

1. choose σ_0 and let $\tau_0 = \tau(\sigma_0)$ (best response)
2. while (σ_k, τ_k) is not optimal, obtain σ_{k+1} by switching σ_k ; let $\tau_{k+1} = \tau(\sigma_{k+1})$

Lemma

For all k, $v_{\sigma_{k+1},\tau_{k+1}} > v_{\sigma_k,\tau_k}$

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The HK algorithm makes at most $O(2^n/n)$ iterations

Computing the value is thus in PLS but the algorithm can take exponential time:

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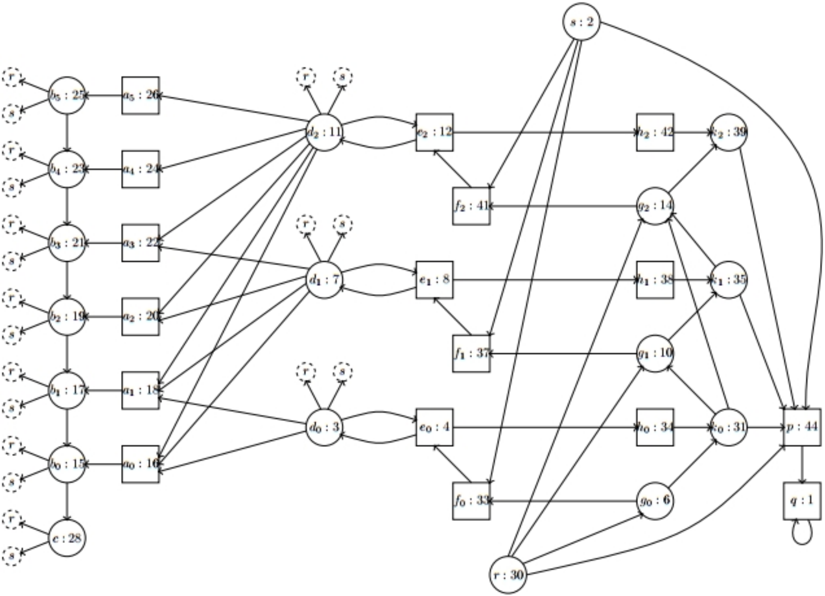
Counter-Example



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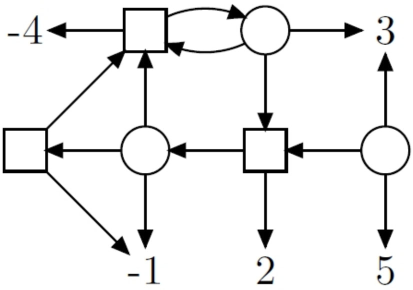

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Definition = SSG without average vertices, but allow sinks with arbitrary payoffs

Solving DGG in linear time by backtracking

While possible :

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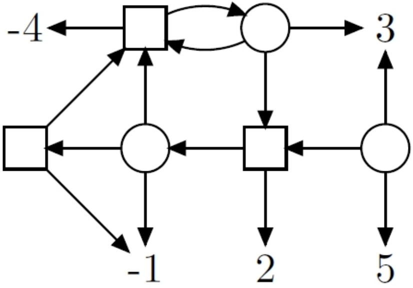

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Almost Acyclic: Tree-Width

Definition (Tree Decomposition)

A tree decomposition of a graph G is a pair (T, X) where $X = \{X_1, \ldots, X_n\}$ is a family of subsets (or bags) of $V(G)$ and T is a tree whose nodes are the X_i such that:

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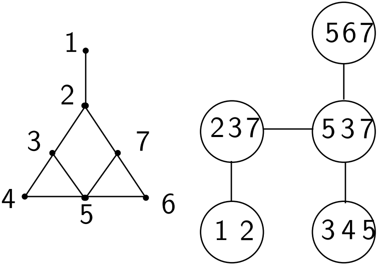

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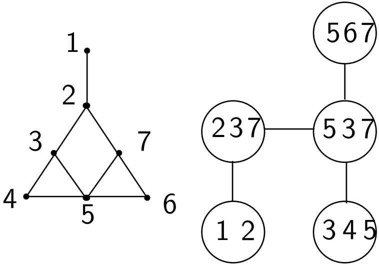

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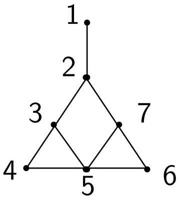

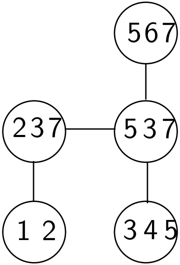

Solving Bounded Treewidth SSGs

Theorem (Work in progress)

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The complexity of the algorithm is in $O(k2^{k^2}n)$.

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