Enumeration of the monomials of a polynomial

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Theory Seminar

Enumeration problems

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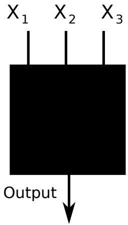

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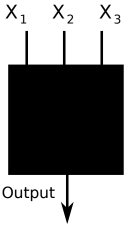

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$$Output = 6$$

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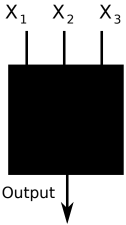

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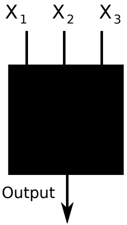

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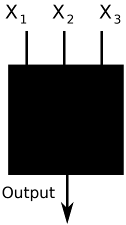

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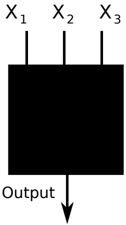

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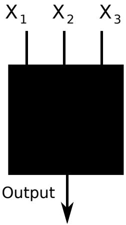

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Let P be a non zero polynomial with n variables of total degree D, if $x_1, \ldots, x_n$ are randomly chosen in a set of integers S of size $\frac{D}{\epsilon}$ then the probability that $P(x_1, \ldots, x_n) = 0$ is bounded by ϵ.

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The *support* of a monomial is the set of indices of variables which appears in the monomial. The support of $X_1 X_3^2 X_5$ is $\{1, 3, 5\}$.

Write P_L for the polynomial P where all variables with indices outside of L set to 0.

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$$P = X_1 X_3^2 X_5 + X_2^4 X_3 + X_1 X_4 + X_2$$

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Building L, an example:

$$P(X_1, X_2, X_3, X_4) = X_1^3 X_2 + X_1 X_3 - 3X_2 X_4 + X_3^2$$

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Theorem

Let P be a polynomial whose monomials have distinct supports with n variables, t monomials and a total degree D. There is an algorithm which computes the set of monomials of P with probability $1-\epsilon$. The delay between the i^{th} and $i+1^{th}$ monomials is bounded by $O(iDn^2(n+\log(\epsilon^{-1})))$ in time and $O(n(n+\log(\epsilon^{-1})))$ calls to the oracle. The algorithm performs $O(tn(n+\log(\epsilon^{-1})))$ calls to the oracle on points of size $\log(2D)$.

Delay: incremental in time and polynomial in the number of calls to the oracle.

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These algorithms are randomized (again the error is exponentially small), and in polynomial time in the number of variables and the degree.

Partial degree

We want to determine the degree of a subset S of variables of the polynomial.

1. pick random values for variables outside of S and look at the remaining polynomial as an univariate one, interpolate it to get its degree
2. evaluate the polynomial on a large value for the variables of S and small random values for the others
3. if the polynomial is given by a circuit, transform it into its homogeneous components with regard to S

These algorithms are randomized (again the error is exponentially small), and in polynomial time in the number of variables and the degree.

Improving the delay

Partial-Monomial
Input: a polynomial given as a black box and two sets of variables L_1 and L_2
Output: accept if there is a monomial in the polynomial in which no variables of L_1 appear, but all of those of L_2 do.

When the polynomial is **multilinear**, this problem can be solved by finding the degree of $P_{\bar{L_1}}$ with regard to L_2: test if the degree is equal to $|L_2|$.

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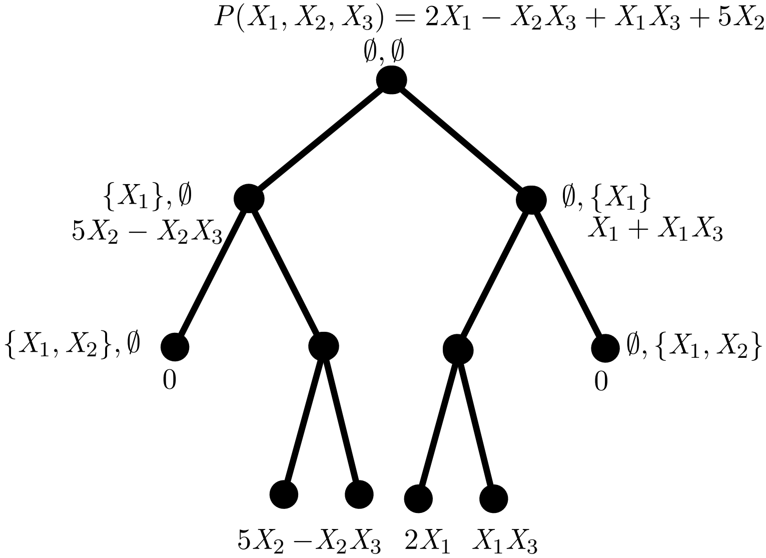
$P(X_1, X_2, X_3) = 2X_1 - X_2X_3 + X_1X_3 + 5X_2$
$\emptyset, \emptyset$
$\{X_1\}, \emptyset$
$5X_2 - X_2X_3$
$\emptyset, \{X_1\}$
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X_1X_3

Polynomial delay algorithm

Theorem

Let P be a multilinear polynomial with n variables and a total degree D. There is an algorithm which computes the set of monomials of P with probability $1 - \epsilon$ and a delay ***polynomial*** *in n, D and $\log(\epsilon)^{-1}$.*

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Comparison to other algorithms

	Ben-Or Tiwari	Zippel	KS	My Algorithm
Algorithm type	Deterministic	Probabilistic	Probabilistic	Probabilistic
Number of calls	$2T$	tnD	tn^7D^4	$tnD(n + \log(\epsilon^{-1}))$
Total time	Quadratic in T	Quadratic in t	Quadratic in t	Linear in t
Enumeration	Exponential	**TotalPP**	**IncPP**	**DelayPP**
Size of points	$T \log(n)$	$\log(nT^2\epsilon^{-1})$	$\log(nD\epsilon^{-1})$	$\log(D)$

Figure: Comparison of interpolation algorithms on multilinear polynomials

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Proposition

The problem MONOMIAL-COEFFICIENT *is* $\#\mathrm{P}$*-hard.*

Proof.

$$Q(X, Y) = \prod_{i=1}^{n} (\sum_{j=1}^{n} X_{i,j} Y_j)$$

The term $T = \prod_{j=1}^{n} Y_j$ has $\sum_{\sigma \in \Sigma_n} \prod_{i=1}^{n} X_{i,\sigma(i)}$ for coefficient, which is the Permanent in the variables $X_{i,j}$. □

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Thank for listening!

Shameless self-promotion

I am a new Post-doc here, working with Pascal Koiran and Natacha Portier.
Interested to work in complexity in general and especially:

- decomposition of matroids, hypergraphs and other structures (width notions)
- circuit complexity
- enumeration complexity
- implicit complexity