Enumeration: logic and algebric methods

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Enumeration problem defined by a formula

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A formula for independent sets:

$$IS(T) \equiv \forall x \forall y \; T(x) \wedge T(y) \Rightarrow \neg E(x, y).$$

The formula is in Π_1, thus $\textsc{Enum}{\cdot}\mathrm{IS} \in \textsc{Enum}{\cdot}\Pi_1$.

Example

The hitting sets (vertex covers) of an hypergraph. An hypergraph H is represented by the incidence structure $\langle D, \{V, E, R\}\rangle$, D is partitioned into V (vertices) and E (edges), R is the incidence relation.

$$HS(T) \equiv \forall x\,(T(x) \Rightarrow V(x)) \wedge \forall y \exists x\, E(y) \Rightarrow (T(x) \wedge R(x, y))$$

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Theorem (Durand, Grandjean 2007)

Let $\varphi(\mathbf{x})$ be a formula of the first order logic over bounded-degree structures. Then $\textsc{Enum}{\cdot}\varphi$ can be enumerated after a linear preprocessing with constant delay.

Theorem (Courcelle 2009)

Let $\varphi(\mathbf{x}, \mathbf{T})$ be a formula of the monadic second order logic over trees, or relational structure of tree-width at most k. Then $\textsc{Enum}{\cdot}\varphi$ can be enumerated after a preprocessing that takes time $O(n.log(n))$ and with delay $O(n)$, where n is the number of vertices or elements.

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A hierarchy of quantifier alternations

From a formula $\Phi(\mathbf{z}, \mathbf{T})$, one defines the counting function $\#\Phi$ which to a model $\mathcal{S}$ associates $|\Phi(\mathcal{S})|$.

Theorem (Saluja, Thakur 1995)

On linearly ordered structures, we have the following inclusions:
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From first order to propositional logic

- a first order formula $\Phi(\mathbf{z}, T)$, $|z| = k$ and $ar(T) = r$.
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Not possible to improve the preprocessing: computing if there is a k-clique should depend on k.

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Union: elimination of some existential quantifiers

Proposition

Let R and S be two polynomially balanced predicates such that S can be decided in time $O(h(n))$. Assume that one can solve $\textsc{Enum}\cdot R$ and $\textsc{Enum}\cdot S$ with preprocessing $f(n)$ and delay $g(n)$, then one can solve $\textsc{Enum}\cdot R \cup S$ with preprocessing $2f(n) + c$ and delay $2g(n) + h(n) + c$, where c is a constant.

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- Somewhat equivalent to the enumeration of the models of a DNF formula with n variables and clauses of fixed size.

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- Somewhat equivalent to the enumeration of the models of a DNF formula with n variables and clauses of fixed size.

Tractable fragments

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Unless $\mathrm{P} = \mathrm{NP}$*, there is no polynomial delay algorithm for* $\textsc{Enum}\cdot\Pi_1$.

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The problem $\textsc{Enum}\cdot\mathrm{SAT}(\mathcal{C})$ *is in* $\textsc{Delay}\mathrm{P}$ *when* $\mathcal{C}$ *is one of the following classes: Horn formulas, anti-Horn formulas, affine formulas, bijunctive (2CNF) formulas*

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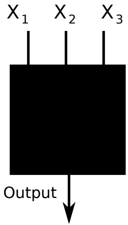

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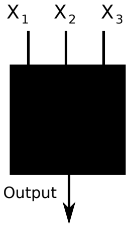

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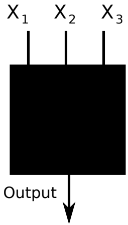

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$$-1 * 1 + -1 * 2 + 1 + 2$$

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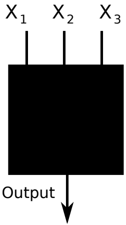

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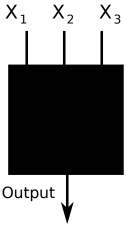

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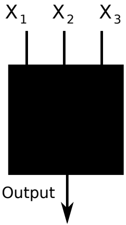

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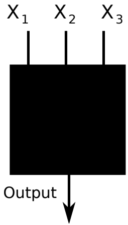

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Polynomial Identity Testing
Input: a polynomial given as a black box.
Output: decides if the polynomial is zero.

Lemma (Schwarz-Zippel)

Let P be a non zero polynomial with n variables of total degree D, if $x_1, \dots, x_n$ are randomly chosen in a set of integers S of size $\frac{D}{\epsilon}$ then the probability that $P(x_1, \dots, x_n) = 0$ is bounded by ϵ.

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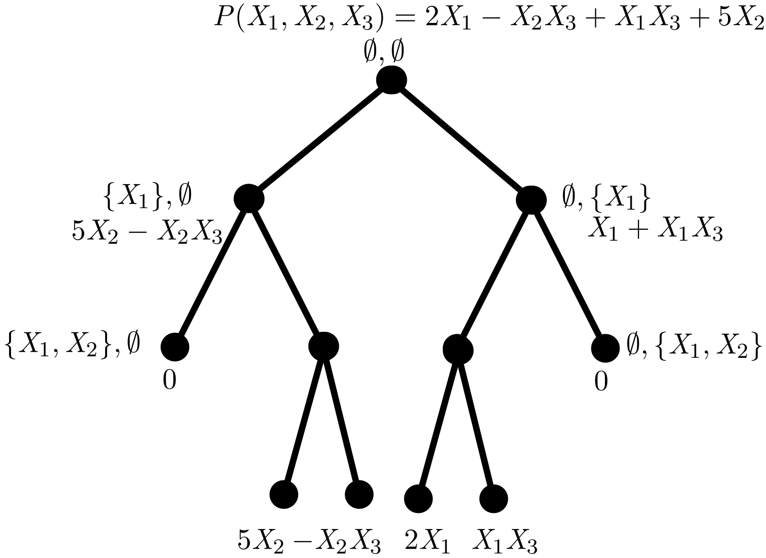
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*Let P be a multilinear polynomial with n variables and a total degree D. There is an algorithm which computes the set of monomials of P with probability $1 - \epsilon$ and a delay **polynomial** in n, D and $\log(\epsilon)^{-1}$.*

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Number of calls	$2T$	tnD	tn^7D^4	$tnD(n + \log(\epsilon^{-1}))$
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