Hypergraph decomposition

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Equipe ALGO

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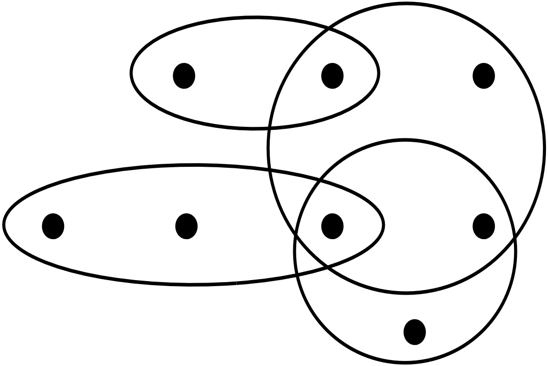
Hypergraphs



Hard problems:

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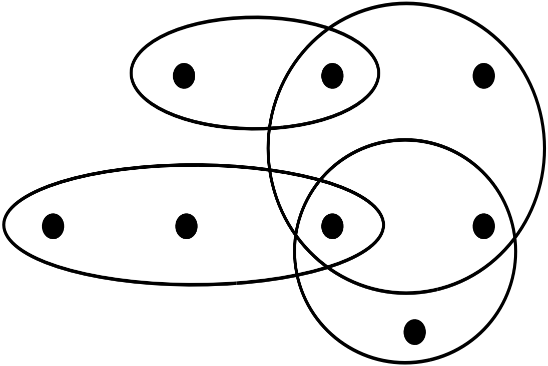

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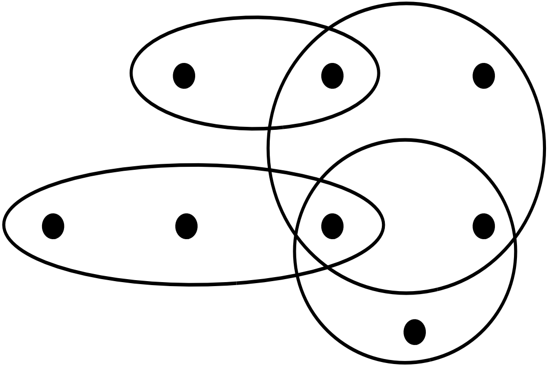

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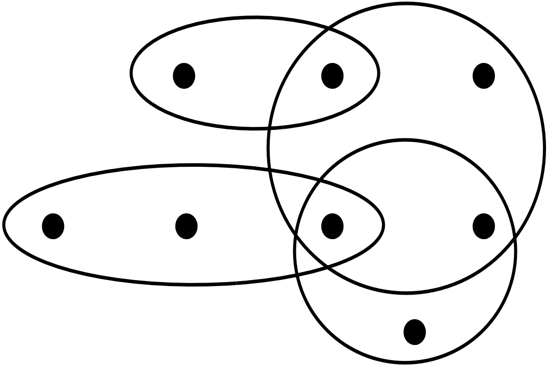

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Simple idea, use a graph decomposition : treewidth of a hypergraph.

Alternatively transform the hypergraph into a graph.

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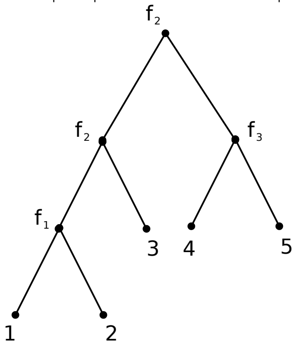

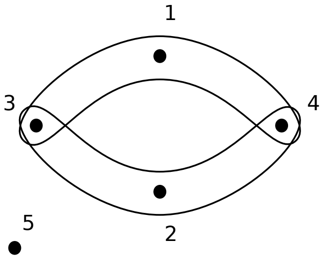

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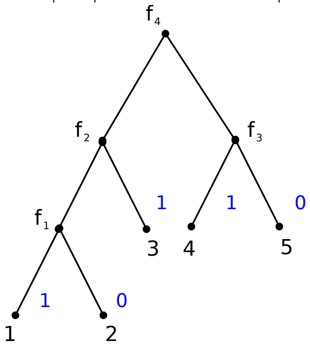

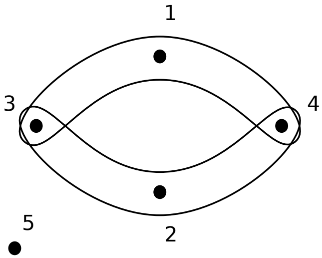

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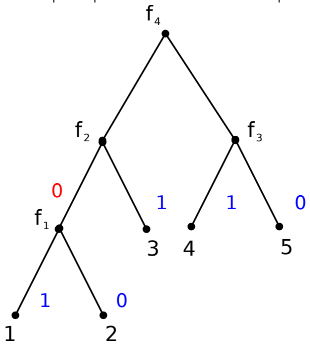

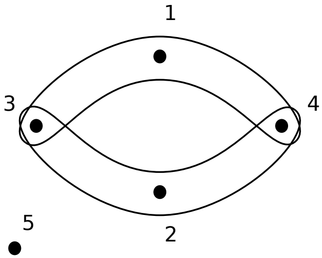

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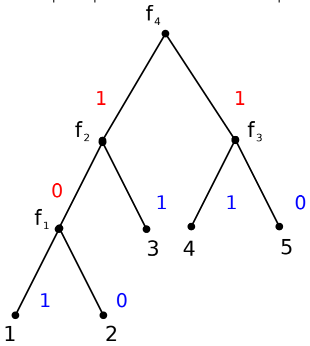

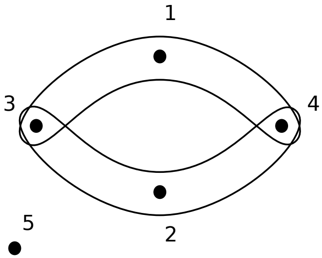

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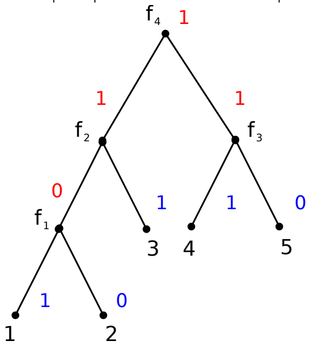

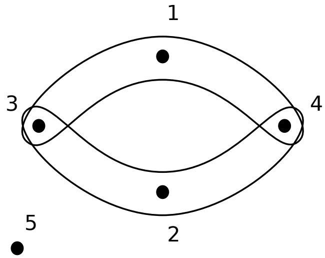

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F_t is the set of unary and binary functions of domain and codomain $\{0, \ldots, t\}$.

$\mathcal{H}_t$ the set of terms $T(F_t, \{0, \ldots, t\})$.

Definition

Let T be a term of $\mathcal{H}_t$, let L be the set of leaves of T and let X be one of its subsets. The value of the term T where the labels of the leaves not in X are replaced by 0 is called the value of X in T and is denoted by $v(X, T)$.

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Related to decomposition notions of matroids, given by their sets of circuits.

- Decomposition-width of matroids [Kral].
- Branch-width of representable matroids [Hlineny].
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Quantification over vertices and set of vertices.

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Let φ be a MSO formula of size l and H a hypergraph with n vertices given by a term of $\mathcal{H}_t$. There is an algorithm which decides whether $H \models \varphi$ in time $f(t, l) \times n$ where f is a computable function.

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Bounds on the decomposition-width

Proposition

Let H be a hypergraph with n vertices, then $\mathrm{dw}(H) \leq 2^{\lceil \frac{n}{2} \rceil}$.

Idea of proof: choose a partition of the vertices into two equal parts. Build a term for each with one color for each hyperedge.

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Let $T \in \mathcal{H}_t$ be a term which represents the hypergraph H and let T' be one of its subterm. Let L be the set of vertices in T' then $|\text{Type}(L)| \leq t + 1$.

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Explicit family of large decomposition-width

First family: $H_{k,n} = ([n], \{X \mid |X| = k\})$

Proposition

For all $n > 3k$, we have $\mathrm{dw}(H_{k,n}) = k + 1$.

Idea of proof : for any decomposition T of $H_{k,n}$, find a subterm whose set of leaves L satisfies $n/3 \leq |L| \leq 2n/3$. The type of a set is roughly its size. Thus $\mathrm{Type}(L) = k + 1$.

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A decomposition adapted to k-uniform hypergraphs:

$D = \{(0,0),(k,0),(k+1,0)\} \cup \{(i,j)\}_{0<i<k, 0\leq j\leq t}$
$\mathcal{F}_{k,t}$ is the set of unary and binary functions with domain and codomain D which satisfy for all $(a,b),(c,d) \in D^2$:

- $f((a,b)) = (a,c)$ for some $c \leq t$
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Uniform decomposition-width: the smallest t such that H is represented by a term of $\mathcal{H}_{k,t}$ denoted by $\mathrm{dw}_u(H)$.

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All hypergraphs represented by a term of $\mathcal{H}_{k,t}$ are k-uniform and the following holds:

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