# Enumeration: logic, algebraic and geometric methods

Yann Strozecki

Université Paris Sud - Paris $11$
Equipe ALGO

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# Enumeration problems

Polynomially balanced predicate $A(x, y)$, decidable in polynomial time.

- $\exists ? y A(x, y)$ : decision problem (class $\mathrm{NP}$)
- $\sharp\{y \mid A(x, y)\}$ : counting problem (class $\sharp\mathrm{P}$)
- $\{y \mid A(x, y)\}$ : enumeration problem (class $\mathrm{EnumP}$)

Example

**Perfect matching:**

- The decision problem is to decide if there is a perfect matching.
- The counting problem is to count the number of perfect matchings.
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# Time complexity measures for enumeration

1. the total time related to the number of solutions
   - polynomial total time: $\mathbf{TotalP}$
2. the delay
   - incremental polynomial time: $\mathbf{IncP}$ (Circuits of a matroid)
   - polynomial delay: $\mathbf{DelayP}$ (Perfect Matching [Uno])
   - Constant or linear delay
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$R$: polynomially balanced binary predicate

$\textsc{Enum}{\cdot}R$

*Input:* $x \in \mathcal{I}$

*Output:* an enumeration of elements in $R(x) = \{y \mid R(x,y)\}$

**Definition**

The problem $\textsc{Enum}{\cdot}R$ belongs to the class $\textsc{Delay}(g,f)$ if there exists an enumeration algorithm that computes $\textsc{Enum}{\cdot}R$ such that, for all input $x$:

- Preprocessing in time $O(g(|x|))$,
- Solutions $y \in R(x)$ are computed successively without repetition with a delay $O(f(|x|))$

$\textsc{Constant-Delay} = \bigcup_k \textsc{Delay}(n^k, 1)$.

# Enumeration complexity classes

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# Of polynomials and automa

Probabilistic Automaton $A = (n, \Sigma, M, \alpha, \eta)$.

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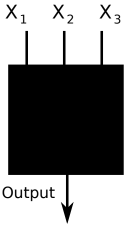


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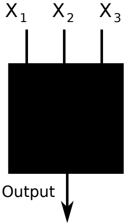


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$$1 * 2 + 1 * 1 + 2 + 1$$

$$Output = 6$$

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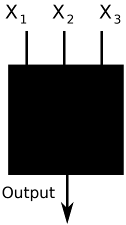


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$$X_1 = -1,\ X_2 = 1,\ X_3 = 2$$

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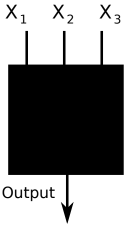


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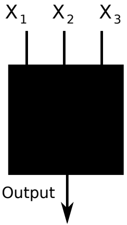


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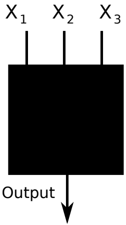


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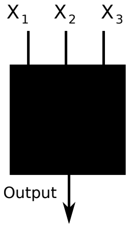


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POLYNOMIAL IDENTITY TESTING
*Input:* a polynomial given as a black box.
*Output:* decides if the polynomial is zero.

**Lemma (Schwarz-Zippel)**

*Let $P$ be a non zero polynomial with $n$ variables of total degree $D$, if $x_1, \ldots, x_n$ are randomly chosen in a set of integers $S$ of size $\frac{D}{\epsilon}$ then the probability that $P(x_1, \ldots, x_n) = 0$ is bounded by $\epsilon$.*

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Partial-Monomial
*Input:* a polynomial given as a black box and two sets of variables $L_1$ and $L_2$
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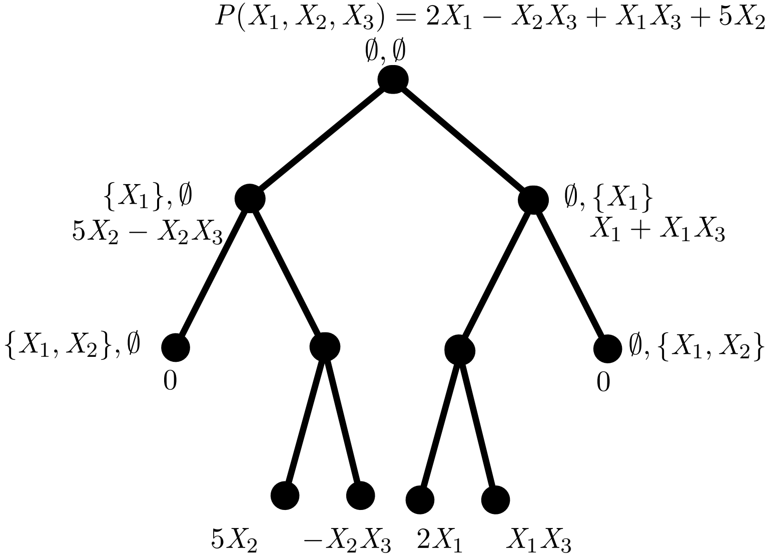
# Depth-first traversal of the monomial tree



# Polynomial delay algorithm

**Theorem**

*Let $P$ be a multilinear polynomial with $n$ variables. There is an algorithm which computes the set of monomials of $P$ with probability $1 - \epsilon$ and a delay **polynomial** in $n$ and $\log(\epsilon)^{-1}$.*

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# Comparison to other algorithms

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| Algorithm type | Deterministic | Probabilistic | Probabilistic | Probabilistic |
| Number of calls | $2T$ | $tnD$ | $tn^7D^4$ | $tnD(n + \log(\epsilon^{-1}))$ |
| Total time | Quadratic in $T$ | Quadratic in $t$ | Quadratic in $t$ | Linear in $t$ |
| Enumeration | Exponential | **TotalPP** | **IncPP** | **DelayPP** |
| Size of points | $T\log(n)$ | $\log(nT^2\epsilon^{-1})$ | $\log(nD\epsilon^{-1})$ | $\log(D)$ |

Figure: Comparison of interpolation algorithms on multilinear polynomials

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**Proposition**

*The problem* Partial-Monomial *restricted to degree* $2$ *polynomials is* $\mathrm{NP}$*-hard.*

# Limits to efficient interpolation

**Strategy:** relate the enumeration problem to some decision problem.

PARTIAL-MONOMIAL
*Input:* a polynomial given as a black box and two sets of variables $L_1$ and $L_2$
*Output:* accept if there is a monomial in the polynomial in which no variables of $L_1$ appear, but all of those of $L_2$ do.

The polynomial delay algorithm works by repeatedly solving this problem.

**Proposition**

*The problem* PARTIAL-MONOMIAL *restricted to degree* $2$ *polynomials is* NP*-hard.*

Introduction to Enumeration

Enumeration and polynomials

Enumeration and logic

Enumeration and polytopes

# Logic in half a slide

**First order logic(FO):**

- Variables: $x, y, z \ldots$
- The language $\sigma$, relations and functions: $R(x, y)$, $f(z)$
- Unary and binary connectors: $\wedge$, $\vee$, $\neg$
- Quantifiers: $\forall$, $\exists$
- $\varphi \equiv \forall x \exists y E(x, y) \vee E(y, x)$

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# Enumeration problem defined by a formula

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*Input:* A $\sigma$-structure $\mathcal{S}$
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# Example

## Example

Independent sets:

$$IS(T) \equiv \forall x \forall y \;\; T(x) \wedge T(y) \Rightarrow \neg E(x, y).$$

## Example

Hitting sets (vertex covers) of a hypergraph represented by the incidence structure $\langle D, \{ V, E, R \} \rangle$.

$$HS(T) \equiv \forall x \, ( T(x) \Rightarrow \; V(x)) \wedge \forall y \exists x \, E(y) \Rightarrow ( T(x) \wedge R(x, y))$$

# First-order queries with free second order variables

**This presentation**

- **FO** queries with free second-order variables
- Data complexity: the query is fixed
- The complexity in term of the size of the input structure's domain
- Quantifier depth as a parameter: $\textsc{Enum}\cdot\Sigma_1$
- $\textsc{Enum}\cdot\mathrm{IS} \in \textsc{Enum}\cdot\Pi_1$ and $\textsc{Enum}\cdot\mathrm{HS} \in \textsc{Enum}\cdot\Pi_2$

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# Previous results

1. Only first-order free variables and bounded degree structures. Durand-Grandjean'07, Lindell'08, Kazana-Segoufin'10: linear preprocessing + constant delay.
2. Only first-order free variables and acyclic conjunctive formula. Bagan-Durand-Grandjean'07: linear preprocessing + linear delay

**Example**

Enumeration of the $k$-cliques of a graph of bounded degree.

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**Example**

Typical database query. Simple paths of length $k$.

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**Example**

Enumeration of the cliques of a bounded tree-width graph.

# A hierarchy result for counting functions

From a formula $\Phi(\mathbf{z}, \mathbf{T})$, one defines the counting function:

$$\#\Phi : \mathcal{S} \mapsto |\Phi(\mathcal{S})|.$$

**Theorem (Saluja, Subrahmanyam, Thakur 1995)**

*On linearly ordered structures:*
$\#\Sigma_0 \subsetneq \#\Sigma_1 \subsetneq \#\Pi_1 \subsetneq \#\Sigma_2 \subsetneq \#\Pi_2 = \sharp\mathrm{P}$.

Some $\sharp\mathrm{P}$-hard problems in $\#\Sigma_1$ (but existence of FPRAS at this level).

**Corollary**

*On linearly ordered structures:*
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# The first level: Enum$\cdot\Sigma_0$

**Theorem**

*For $\varphi \in \Sigma_0$, $\textsc{Enum}\cdot\varphi$ can be enumerated with preprocessing $O(|D|^k)$ and delay $O(1)$ where $k$ is the number of free first order variables of $\varphi$ and $D$ is the domain of the input structure.*

**Simple ingredients:**

1. Transformation of a f.o. formula $\Phi(\mathbf{z}, T)$ into a propositional formulas for each $\mathbf{z}$.
2. Solve the propositional formula to obtain a minimal solution for $T$.
3. Gray Code Enumeration to extend $T$.

# Bounded degree structure

**Remark:** The $k$-clique query is definable.
No hope to improve the $O(|D|^k)$ preprocessing.

**Theorem**

*Let $d \in \mathbb{N}$, on $d$-degree bounded input structures, $\textsc{Enum}\cdot\Sigma_0 \in \textsc{Delay}(|D|, 1)$ where $D$ is the domain of the input structure.*

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**Idea of proof:**

- Another transformation: $\Phi(\mathbf{z}, T)$ seen as a propositional formula whose variables are the atoms of $\Phi$.
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# Second level: Enum$\cdot\Sigma_1$

**Theorem**

$\textsc{Enum}\cdot\Sigma_1 \subseteq \textsc{DelayP}$. *More precisely,* $\textsc{Enum}\cdot\Sigma_1$ *can be computed with precomputation* $O(|D|^{h+k})$ *and delay* $O(|D|^k)$ *where* $h$ *is the number of free first order variables of the formula,* $k$ *the number of existentially quantified variables and* $D$ *is the domain of the input structure.*

**Idea of Proof:** $\Phi(\mathbf{y}, T) = \exists \mathbf{x} \varphi(\mathbf{x}, \mathbf{y}, T)$

- Substitute values for $\mathbf{x}$. Collection of formulas of the form:

$$\varphi(\mathbf{x}^*, \mathbf{y}, T)$$

- Need to enumerate the union.

# Enumeration of an union

Enumerate the solution of $\text{ENUM}\cdot R$ and $\text{ENUM}\cdot S$.

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# The case Enum$\cdot\Pi_1$

**Proposition**

*Unless* $\mathrm{P} = \mathrm{NP}$, *there is no polynomial delay algorithm for* $\textsc{Enum}\cdot\Pi_1$.

**Proof** Direct encoding of SAT.

Hardness even:

- on the class of bounded degree structure
- if all clauses but one have at most two occurences of a second-order free variable

# Tractable cases

Problem $\textsc{Enum}{\cdot}\Phi$ with $\Phi \in \Sigma_i$: transformation of $\Phi$ into a propositional formula $\tilde{\Phi}$.

**Corollary**

*Let $\Phi(\mathbf{z}, T)$ be a formula, such that, for all $\sigma$ structures, all propositional formulas $\tilde{\Phi}$ are either Horn, anti-Horn, affine or bijunctive. Then $\textsc{Enum}{\cdot}\Phi \subseteq \textsc{Delay}\mathrm{P}$.*

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The polynomial for the automaton $A$:

$$\sum_{w \in \Sigma^n} A(w) X_{1,\sigma_1} X_{2,\sigma_2} \ldots X_{n,\sigma_n}$$

Possible to change $n$, the size of the generated words.

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# The polytope separation

Let $K_1, K_2$ be two polytopes. We want to produce a point in $K_1 \triangle K_2$.
Enumeration = **uniform sampling**.

Different representation of polytopes:

1. Convex hull of a set of given points: $\mathcal{V}$-polytope
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Negate a constraint of $K_1$, add it to $K_2$ then decide whether the system has a solution.

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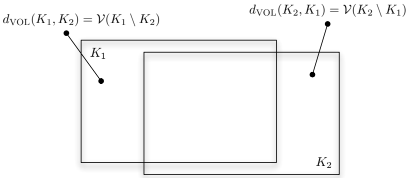
# Distance and random walk



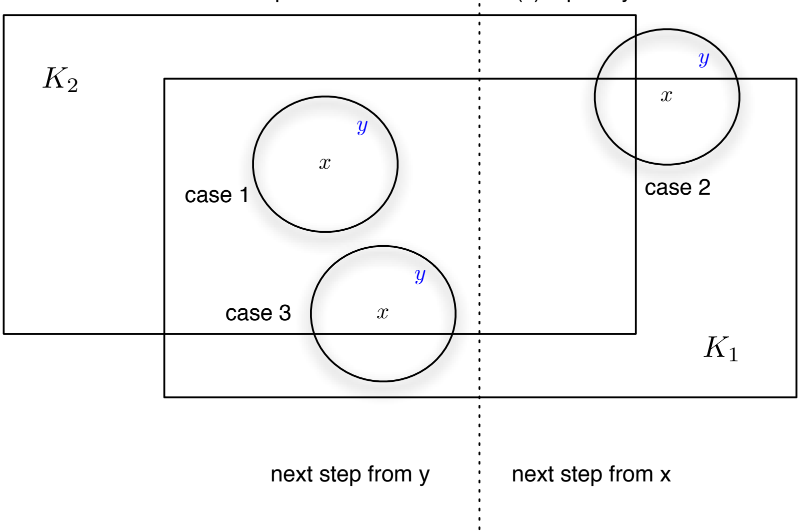

$$d_{vol}(K_1, K_2) = \max\left\{\frac{d_{\mathrm{VOL}}(K_1, K_2)}{\mathcal{V}(K_1)}, \frac{d_{\mathrm{VOL}}(K_2, K_1)}{\mathcal{V}(K_2)}\right\}$$

**Theorem**

*Let $K_1$ and $K_2$ be two polytopes, given as SMOs. For all $\epsilon > 0$, if $d_{vol}(K_1, K_2) \geq \varepsilon$, then the* Polytope Separator *outputs a point $x$ in $K_1 \triangle K_2$ with probability greater than $2/3$. Moreover, the running time of this algorithm is polynomial in $n$ and $\epsilon^{-1}$.*

# The ball walk

We use the Ball Walk... we pick at random unif. in B(x) a point y : 3 cases



# Approximate verification: witness generation

**Theorem**

*Given two regular expressions $r_1, r_2$ on words and $\epsilon$, if $d_{vol}(H_{r_1}, H_{r_2}) \geq \lambda$, we can generate $\epsilon$-separating words in polynomial time in the dimension and $1/\lambda$.*

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Let's all do enumeration