The Limited Power of Powering: Polynomial Identity Testing and a Depth-four Lower Bound for the Permanent

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────── **Abstract** ──────

Polynomial identity testing and arithmetic circuit lower bounds are two central questions in algebraic complexity theory. It is an intriguing fact that these questions are actually related. One of the authors of the present paper has recently proposed a "real τ-conjecture" which is inspired by this connection. The real τ-conjecture states that the number of real roots of a sum of products of sparse univariate polynomials should be polynomially bounded. It implies a superpolynomial lower bound on the size of arithmetic circuits computing the permanent polynomial.

In this paper we show that the real τ-conjecture holds true for a restricted class of sums of products of sparse polynomials. This result yields lower bounds for a restricted class of depth-4 circuits: we show that polynomial size circuits from this class cannot compute the permanent, and we also give a deterministic polynomial identity testing algorithm for the same class of circuits.

1   Introduction

The τ-conjecture [18, 19] states that a univariate polynomial with integer coefficients defined by an arithmetic circuit has a number of integer roots polynomial in the size of the circuit. A real version of this conjecture was recently presented in [14]. The real τ-conjecture states that the number of real roots of a sum of products of sparse univariate polynomials should be polynomially bounded as a function of the size of the corresponding expression. More precisely, consider a polynomial of the form

$$f(X) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{ij}(X),$$

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where $f_{ij} \in \mathbb{R}[X]$ has at most t monomials. The conjecture asserts that the number of real roots of f is bounded by a polynomial function of kmt. It was shown in [14] that this conjecture implies a superpolynomial lower bound on the arithmetic circuit complexity of the permanent polynomial (a central goal of algebraic complexity theory ever since Valiant's seminal work [20]). In this paper we show that the conjecture holds true in a special case. We focus on the case where the number of distinct sparse polynomials is small (but each polynomial may be repeated many times). We therefore consider expressions of the form

$$\sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}}(X). \tag{1}$$

We obtain a $O(t^{m(2^{k-1}-1)})$ upper bound on the number of real roots of such a polynomial, where t is the maximum number of monomials in the f_j's. In particular, the bound is polynomial in t when the "top fan-in" k and the number m of sparse polynomials in the expression are both constant. Note also that the bound is independent of the magnitude of the integers α_{ij}.

From this upper bound we obtain a lower bound on the complexity of the permanent for a restricted class of arithmetic circuits. The circuits that we consider are again of form (1), but now X should be interpreted as the tuple of inputs to the circuit rather than as a single real variable. Roughly speaking, we show a superpolynomial lower bound on the complexity of the permanent in the case where k and m are again fixed. More precisely, we show that a circuit of form (1) cannot compute the permanent as long as the sparsity of the f_j's is polynomially bounded. Note that this is a lower bound for a restricted class of depth-4 circuits: The output gate at depth 4 has fan-in bounded by the constant k, and the gates at depth 2 are only allowed to compute a constant (m) number of distinct polynomials f_j.

Our third main result is a deterministic identity testing algorithm, again for polynomials of the same form. When k and m are fixed, we can test if the polynomial in (1) is identically equal to 0 in time polynomial in t and in $\max_{ij} \alpha_{ij}$. Note that if k, *m and the exponents* α_{ij} are all bounded by a constant then the number of monomials in such a polynomial is $t^{O(1)}$ and our three main results become trivial. These results are therefore interesting only in the case where the α_{ij} may be large, and can be interpreted as limits on the power of powering.

1.1   Connection to Previous Work

The idea of deriving lower bounds on arithmetic circuit complexity from upper bounds on the number of real roots goes back at least to a 1976 paper by Borodin and Cook [6]. Their results were independently improved by Grigoriev and Risler (see [8], chapter 12). For a long time, it seemed that the lower bounds that can be obtained by this method had to be rather small since the number of real roots of a polynomial can be exponential in its arithmetic circuit size. Nevertheless, as explained above it was recently shown in [14] that superpolynomial lower bounds on the complexity of the permanent on general arithmetic circuits can be derived from a suitable upper bound on the number of roots of sums of products of sparse polynomials. This is related to the fact that for low degree polynomials, arithmetic circuits of depth 4 are almost equivalent to general arithmetic circuits [3, 13].

The study of polynomial identity testing (PIT) also has a long history. The Schwartz-Zippel lemma [17] yields a randomized algorithm for PIT. A connection between deterministic PIT and arithmetic circuit lower bounds was pointed out as early as 1980 by Heintz and Schnorr [11], but a more in-depth study of this connection began only much later [12]. The recent literature contains deterministic PIT algorithms for various restricted models (see

e.g. the two surveys [2, 16]). These algorithms are either black-box, i.e. the algorithm can only test the circuit for zero on inputs of its own choosing, or non-black-box in which case the algorithm has access to the structure of the circuit. One model which is similar to ours was recently studied in [5]. It follows from Theorem 1 in [5] that there is a polynomial time deterministic black-box PIT algorithm for polynomials of the form (1) if, instead of bounding k and m as in our algorithm, we bound the transcendence degree r of the polynomials f_j. Obviously we have $r \leq m$, so from this point of view their result is more general.[1] On the other hand their running time is polynomial in the degree of the f_j, whereas we can handle polynomials of exponential degree in polynomial time. Furthermore they not provide any lower bound result for the permanent. (Note that the bound that is deduced from their black-box PIT algorithm using the technique of [1] only applies to polynomial families with coefficients in PSPACE and not to the permanent family.)

1.2 Our approach

The proof of our bound on the number of real roots has the same high-level structure as that of Descartes' rule of signs.

▶ **Proposition 1.** *A univariate polynomial $f \in \mathbb{R}[X]$ with $t \geq 1$ monomials has at most $t-1$ positive real roots.*

The number of negative roots of f is also bounded by $t-1$ (consider $f(-X)$), hence there are at most $2t-1$ real roots (including 0). There is also a refined version of Proposition 1 where the number of monomials t is replaced by the number of sign changes in the sequence of coefficients of f. The cruder version will be sufficient for our purposes.

We briefly recall an inductive proof of Proposition 1. For $t = 1$, there is no non-zero root. For $t > 1$, let $a_\alpha X^\alpha$ be the monomial of lowest degree. We can assume that $\alpha = 0$ (if not, we can divide f by X^α since this operation does not change the number of positive roots). Consider now the derivative f'. It has $t-1$ monomials, and at most $t-2$ positive real roots by induction hypothesis. Moreover, by Rolle's theorem there is a positive root of f' between 2 consecutive positive roots of f. We conclude that f has at most $(t-2)+1 = t-1$ positive roots.

In (1) we have a sum of k terms instead of t monomials, but the basic strategy remains the same: we divide by the first term and take the derivative. This has the effect of removing a term, but it also has the effect (unlike Descartes' rule) of increasing the complexity of the remaining $k-1$ terms. This results in a larger bound (and a longer proof).

From this upper bound we obtain our permanent lower bound by applying the proof method which was put forward in [14]. More precisely, assume that the permanent has an efficient representation of the form (1). We show that the same must be true for the univariate polynomial $\prod_{i=1}^{2^n}(X-i)$ using a result of Bürgisser [7]. This yields a contradiction with our upper bound on the number of real roots.

Our third result is a polynomial identity testing algorithm. Using a standard substitution technique, we can assume that the polynomials f_j in (1) are univariate. We note that the resulting f_j may be of exponential degree even if the original multivariate f_j are of low degree. The construction of hitting sets is a classical approach to deterministic identity testing. Recall that a hitting set for a class $\mathcal{F}$ of polynomials is a set of points H such that for any non-identically zero polynomial $f \in \mathcal{F}$ we have a point $x \in H$ such that $f(x) \neq 0$.

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[1] As pointed out by the authors of [5], their result already seems nontrivial for a constant m.

Clearly, a hitting set yields a black-box identity testing algorithm (it is not hard to see that the converse is also true). Moreover, for any class $\mathcal{F}$ of univariate polynomials, an upper bound $z(\mathcal{F})$ on the number of real roots of each non-zero polynomial in $\mathcal{F}$ yields a hitting set (any set of $z(\mathcal{F}) + 1$ real numbers will do). From our upper bound result we therefore have polynomial size hitting sets for polynomials of the form (1) when k and m are fixed. Unfortunately, the resulting black-box algorithm does not run in polynomial time: evaluating a polynomial at a point of the hitting set may not be feasible in polynomial time since (as explained above) the f_j may be of very high degree. We therefore use a different strategy. Roughly speaking, we "run" the proof of our upper bound theorem on an input of form (1). This requires explicit knowledge of this representation, and the resulting algorithm is non-black-box. As explained in Section 1.1, for the case where the f_j are low-degree multivariate polynomials an efficient black-box algorithm was recently given in [5].

Organization of the paper. In Section 2 we prove an upper bound on the number of real roots of polynomials of the form (1), see Theorems 10 and 11 at the end of the section. In fact, we obtain an upper bound for a more general class of polynomials which we call $\mathsf{SPS}(k, m, t, h)$. This generalization is needed for the inductive proof to go through. From this upper bound, we derive in Section 3 a lower bound on the computational power of (multivariate) circuits of the same form. We give in Section 4 a deterministic identity testing algorithm, again for polynomials of form (1).

2 The real roots of a sum of products of sparse polynomials

2.1 Definitions

In this section, we define precisely the polynomials we are working with. We then explain how to transform those polynomials in a way which reduces the number of terms but does not increase too much the number of roots. This method has some similarities with the proof of Lemma 2 in [15] and it leads to a bound on the number of roots of the polynomials we study.

We say that a polynomial is *t-sparse* if it has at most t monomials.

▶ **Definition 2.** Let $\mathsf{SPS}(k, m, t, h)$ denote the class of polynomials $\phi \in \mathbb{R}[X]$ defined by

$$\phi(X) = \sum_{i=1}^{k} g_i(X) \prod_{j=1}^{m} f_j^{\alpha_{ij}}(X)$$

where
- $g_1, \ldots, g_k$ are h-sparse polynomials over $\mathbb{R}$;
- $f_1, \ldots, f_m$ are t-sparse non-zero polynomials over $\mathbb{R}$;
- $\alpha_{11}, \ldots, \alpha_{km}$ are non-negative integers.

We define $P_i = \prod_{j=1}^{m} f_j^{\alpha_{ij}}$ and $T_i = g_i P_i$ for all i. We also define $\pi = \prod_{j=1}^{m} f_j$. Finally, we define $\mathsf{SPS}(k, m, t)$ as the subclass of $\mathsf{SPS}(k, m, t, h)$ in which all the g_i are equal to the constant 1.

Note that $\mathsf{SPS}(k, m, t)$ is just the class of polynomials of form (1), and is included in $\mathsf{SPS}(k, m, t, 1)$. We want to give a bound for the number of real roots of the polynomials in this class, and more generally in $\mathsf{SPS}(k, m, t, h)$. To this end, from a polynomial $\phi \in \mathsf{SPS}(k, m, t, h)$, we build in Lemma 4 a new polynomial $\tilde{\phi} \in \mathsf{SPS}(k - 1, m, t, \tilde{h})$ for some $\tilde{h}$ such that a bound on the number of real roots of $\tilde{\phi}$ yields a bound for ϕ. We first give a

bound for the number of roots of the polynomials in $\mathsf{SPS}(2, m, t)$. The proof in this case contains the main ingredients of the general case with less technicalities.

▶ **Proposition 3.** *Let* $\phi = \prod_{j=1}^m f_j^{\alpha_{1j}} + \prod_{j=1}^m f_j^{\alpha_{2j}} \in \mathsf{SPS}(2, m, t)$. *Then* ϕ *has at most* kmt *real roots.*

Proof. Let $\psi = \phi / \prod_j f_j^{\alpha_{1j}} = 1 + \prod_{j=1}^m f_j^{\alpha_{2j} - \alpha_{1j}}$. Then

$$\psi' = \prod_{j=1}^m f_j^{\alpha_{2j} - \alpha_{1j} - 1} \times \sum_{j=1}^m (\alpha_{2j} - \alpha_{1j}) f_j' \prod_{l \neq j} f_l.$$

Since each f_j is t-sparse, the polynomial $\sum_{j=1}^m (\alpha_{2j} - \alpha_{1j}) f_j' \prod_{l \neq j} f_l$ has at most mt^m monomials. Therefore, its number of real roots is at most $2mt^m - 1$ (by Descartes' rule of signs). The number of roots and poles of the rational function $\prod_{j=1}^m f_j^{\alpha_{2j} - \alpha_{1j} - 1}$ is at most $2m(t - 1)$, the total number of roots of the f_j's. Therefore, the number of roots and poles of ψ' is at most $2mt^m - 1 + 2m(t - 1)$. Now, between two consecutive roots of the rational function ψ, there exists a root or a pole of ψ', so there ψ has at most $2m(t^m + t - 1)$ roots. Since the number of roots of ϕ is bounded by the sum of the number of roots of ψ and $\prod_j f_j^{\alpha_{1j}}$, ϕ has at most $2m(t^m + t - 1) + 2m(t - 1)$ real roots. ◀

We now turn to the general case.

▶ **Lemma 4.** *Let* $\phi \in \mathsf{SPS}(k, m, t, h)$. *If* g_1 *is not identically zero, let* $\psi = \phi / T_1$ *and* $\tilde{\phi} = g_1 T_1 \pi \psi'$; *otherwise let* $\tilde{\phi} = \phi$. *Then there exists* $\tilde{h}$ *such that* $\tilde{\phi} \in \mathsf{SPS}(k - 1, m, t, \tilde{h})$.

Proof. If g_1 is identically zero, the theorem holds with $\tilde{h} = h$. Assume now that g_1 is not identically zero. Then

$$\psi(X) = \phi(X) / T_1(X) = 1 + \frac{1}{T_1(X)} \cdot \sum_{i=2}^k T_i(X)$$

and

$$\psi' = \frac{\sum_{i=2}^k (T_1 T_i' - T_1' T_i)}{T_1^2}.$$

Notice that $T_i' = g_i' P_i + g_i P_i'$ and

$$P_i' = \sum_{j=1}^m \alpha_{ij} f_j' f_j^{\alpha_{ij} - 1} \cdot \prod_{l \neq j} f_l^{\alpha_{il}} = P_i \cdot \sum_{j=1}^m \alpha_{ij} f_j' / f_j.$$

Therefore

$$\psi' = \frac{1}{T_1^2} \cdot \sum_{i=2}^k (g_1 P_1 g_i' P_i + g_1 P_1 g_i P_i' - g_1' P_1 g_i P_i - g_1 P_1' g_i P_i)$$

$$= \frac{1}{T_1^2} \cdot \sum_{i=2}^k (g_1 g_i' P_1 P_i + g_1 g_i P_1 P_i \sum_j \alpha_{ij} f_j' / f_j$$

$$\qquad - g_1' g_i P_1 P_i - g_1 g_i P_1 P_i \sum_j \alpha_{1j} f_j' / f_j)$$

$$= \frac{1}{g_1 T_1} \cdot \sum_{i=2}^k P_i \left(g_1 g_i' - g_1' g_i + g_1 g_i \sum_j (\alpha_{ij} - \alpha_{1j}) f_j' / f_j \right).$$

We now multiply ψ' by $\pi = \prod_j f_j$ and get

$$\pi\psi' = \frac{1}{g_1 T_1} \cdot \sum_{i=2}^{k} P_i \left(\pi \cdot (g_1 g_i' - g_1' g_i) + g_1 g_i \sum_j (\alpha_{ij} - \alpha_{1j}) f_j' \prod_{l \neq j} f_l \right).$$

Thus $g_1 T_1 \pi \psi'$ is a polynomial of the class $\mathsf{SPS}(k-1, m, t, \tilde{h})$ for some $\tilde{h}$. Let us write

$$\tilde{\phi} = g_1 T_1 \pi \psi' = \sum_{i=2}^{k} P_i \tilde{g}_i.$$

The integer $\tilde{h}$ denotes the maximum number of monomials in $\tilde{g}_i$ for $2 \leq i \leq k$. ◄

▶ **Definition 5.** Let $(\phi_n)_{1 \leq n \leq k}$ be the sequence defined by $\phi_1 = \phi$ and for $n \geq 1$, $\phi_{n+1} = \tilde{\phi}_n$. Let also, for $1 \leq i \leq k$, $(g_i^{(n)})_{1 \leq n \leq i}$ be such that for $1 \leq n \leq k$,

$$\phi_n = \sum_{i=n}^{k} g_i^{(n)} \prod_{j=1}^{m} f_j^{\alpha_{ij}}.$$

We also define the sequence $(h_n)_{1 \leq n \leq k}$ by $h_1 = 1$ and $h_{n+1} = \tilde{h}_n$. That is, each $g_i^{(n)}$ is h_n-sparse.

2.2   A generalization of Descartes' rule

In Definition 5 we defined a sequence of polynomials (ϕ_n) and a sequence of integers (h_n). In this section we first prove that the number of real roots of ϕ_n is bounded by the number of real roots of ϕ_{n+1} up to a multiplicative constant. Then, we give an upper bound on h_n and we combine these ingredients to obtain a bound on the number of real roots of a polynomial in $\mathsf{SPS}(k, m, t)$. This bound (in Theorem 10 at the end of the section) is polynomial in t.

We denote by $r(P)$ the number of distinct real roots of a rational function P. In order to obtain a bound on $r(\phi)$ from a bound on $r(\tilde{\phi})$, we need the following lemma.

▶ **Lemma 6.** *Let $P \in \mathsf{SPS}(1, m, t, h)$. If P is not identically zero then*

$$r(P) \leq 2h + 2m(t-1) - 1.$$

Proof. By definition, $P = g \cdot \prod_j f_j^{\alpha_j}$. The number of non-zero real roots of P is therefore bounded by the sum of the number of non-zero real roots of g and of the f_j's. Since g is h-sparse, we know from Descartes' rule that is has at most $2(h-1)$ non-zero real roots. Likewise, each f_j has at most $2(t-1)$ real roots. As a result, P has at most $2(h-1) + 2m(t-1)$ non-zero real roots. Since 0 can also be a root, we add 1 to this bound to obtain the final result. ◄

▶ **Lemma 7.** *Let $\phi \in \mathsf{SPS}(k, m, t, h)$. Then*

$$r(\phi) \leq r(\tilde{\phi}) + 4h + 4m(t-1) - 1.$$

Proof. If g_1 is zero in the definition of ϕ, then $\tilde{\phi} = \phi$ which proves the lemma.

Recall from the proof of Lemma 4 the notation $\psi = \phi/T_1$. If g_1 is not identically zero, by definition we have $\tilde{\phi} = g_1 T_1 \pi \psi'$, so the number $r(\tilde{\phi})$ of real roots of the polynomial $\tilde{\phi}$ is an upper bound on the number of real roots of ψ'.

Since $\phi = T_1\psi$, we have $r(\phi) \leq r(T_1) + r(\psi)$. Moreover, between two consecutive roots of the rational function ψ, we have a root of ψ' or a root of the denominator T_1. As a result, $r(\psi) \leq r(\psi') + r(T_1) + 1$. It follows that $r(\phi) \leq r(\psi') + 2r(T_1) + 1 \leq r(\tilde{\phi}) + 2r(T_1) + 1$. Moreover, the polynomial $T_1 = g_1 \cdot \prod_j f_j^{\alpha_{1j}}$ is in $\mathsf{SPS}(1, m, t, h)$. Thus by Lemma 6, T_1 has at most $2h + 2m(t-1) - 1$ real roots. We conclude that ϕ has at most

$$r(\tilde{\phi}) + 2 \cdot (2h + 2m(t-1) - 1) + 1 = r(\tilde{\phi}) + 4h + 4m(t-1) - 1$$

real roots. ◀

▶ **Proposition 8.** *Let $\phi \in \mathsf{SPS}(k, m, t, 1)$. Then*

$$r(\phi) \leq 2h_k + 4\sum_{i=1}^{k-1} h_i + 2m(2k-1)(t-1) - k.$$

Proof. Lemma 7 gives the following recurrence:

$$r(\phi_n) \leq r(\phi_{n+1}) + 4h_n + 4m(t-1) - 1.$$

Thus, we get

$$r(\phi) \leq r(\phi_k) + 4\sum_{i=1}^{k-1} h_i + (k-1)(4m(t-1) - 1). \tag{2}$$

Since $\phi_k \in \mathsf{SPS}(1, m, t, h_k)$, Lemma 6 bounds its number of real roots:

$$r(\phi_k) \leq 2h_k + 2m(t-1) - 1. \tag{3}$$

The bound is a combination of (2) and (3). ◀

Proposition 8 shows that in order to bound $r(\phi)$, we need a bound on h_n.

▶ **Proposition 9.** *For all n, h_n is bounded by $((m+2)t^m)^{2^{n-1}-1}$.*

Proof. As showed in the proof of Lemma 4, $\tilde{\phi} = \sum_{i=2}^{k} \tilde{g}_i P_i$ where each $\tilde{g}_i$ is $\tilde{h}$-sparse. More precisely,

$$\tilde{g}_i = (g_1 g_i' - g_1' g_i) \prod_{j=1}^{m} f_j + g_1 g_i \sum_{j=1}^{m} (\alpha_{ij} - \alpha_{1j}) f_j' \prod_{l \neq j} f_l.$$

Thus $\tilde{g}_i$ is a sum of $(m+2)$ terms, and each term is a product of m t-sparse polynomials by two h-sparse polynomials. Thus $\tilde{h} \leq (m+2)t^m h^2$.

This gives the following recurrence relation on h_n:

$$\begin{cases} h_1 & = 1 \\ h_{n+1} & \leq (m+2)t^m h_n^2 \end{cases}$$

Therefore, $h_n \leq ((m+2)t^m)^{2^{n-1}-1}$. ◀

Now, we combine Propositions 8 and 9 to obtain our first bound on the number of roots of a polynomial in $\mathsf{SPS}(k, m, t)$.

▶ **Theorem 10.** *Let $\phi \in \mathsf{SPS}(k, m, t)$: we have $\phi = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}}$ where for all i and j, f_j is t-sparse and $\alpha_{ij} \geq 0$. Then $r(\phi) \leq C \times ((m+2)t^m)^{2^{k-1}-1}$ for some universal constant C.*

The bound for $\tilde{h}$, the number of monomials in the polynomials $g_i^{(n)}$, can actually be improved. This automatically sharpens the bound we give for the number of real roots of a polynomial in $\mathsf{SPS}(k, m, t)$.

▶ **Theorem 11.** *Let $\phi \in \mathsf{SPS}(k, m, t)$. Then ϕ has at most*

$$C \times \left[e \times \left(1 + \frac{t^m}{2^{k-1} - 1} \right) \right]^{2^{k-1} - 1}$$

real roots, where C is a universal constant.

The proofs of these two theorems can be found in the full version of this paper [10, Theorem 1 and 2].

3   Lower bounds

In this section we introduce a subclass $\mathsf{mSPS}(k, m)$ of the class of "easy to compute" multivariate polynomial families, and we use the results of Section 2.2 to show that it does not contain the permanent family. The polynomials in a $\mathsf{mSPS}(k, m)$ family have the same structure as the univariate polynomials in the class $\mathsf{SPS}(k, m, t)$ from Definition 2. In this section, polynomial families are denoted by their general term in brackets: The polynomial P_n is the n-th polynomial of the family (P_n). When there is no ambiguity on the number of variables, we denote by $\vec{X}$ the tuple of variables of a polynomial P_n. The definition uses the notion of *constant-free* circuit: An arithmetic circuit is said constant-free if the only constant input is -1 (or equivalently are of polynomially bounded bitsize).

▶ **Definition 12.** We say that a sequence of polynomials (P_n) is in $\mathsf{mSPS}(k, m)$ if there is a polynomial Q such that for all n:
(i) P_n depends on at most $Q(n)$ variables.
(ii) $P_n(\vec{X}) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{jn}^{\alpha_{ij}}(\vec{X})$
(iii) The bitsize of α_{ij} is bounded by $Q(n)$.
(iv)   For all $1 \leq j \leq m$, the polynomial f_{jn} has a constant-free circuit of size $Q(n)$ and is $Q(n)$-sparse.

▶ Remark. If $(P_n) \in \mathsf{mSPS}(k, m)$ then each P_n has a constant-free circuit of size polynomial in n. Indeed from the constant-free circuits of the polynomials f_{jn} we can build a constant-free circuit for P_n. We have to take the α_{ij}-th power of f_{jn}, which can be done with a circuit of size polynomial in the bitsize of α_{ij} thanks to fast exponentiation. The size of the final circuit is up to a constant the sum of the sizes of these powering circuits and of the circuits giving f_{jn}, which is thus polynomial in n.

▶ **Definition 13.** The Pochhammer-Wilkinson polynomial of order 2^n is defined by

$$\mathrm{PW}_n = \prod_{i=1}^{2^n} (X - i).$$

▶ **Definition 14.** The Permanent over n^2 variables is defined by $\mathrm{PER}_n = \displaystyle\sum_{\sigma \in \Sigma_n} \prod_{i=1}^{n} X_{i\sigma(i)}$

where Σ_n is the set of permutations of $\{1, \ldots, n\}$.

We now give a lower bound on the Permanent, using its completeness for VNP [20], a result of Bürgisser on the Pochhammer-Wilkinson polynomials [7] and our bound on the roots of the polynomials in $\mathsf{SPS}(k, m, t)$. We refer to Bürgisser's book [9] for the definition and properties of VNP.

▶ **Theorem 15.** *The family of polynomials* (PER_n) *is not in* $\mathsf{mSPS}(k, m)$ *for any fixed* k *and* m, *i.e., there is no representation of the permanent family of the form*

$$\mathrm{PER}_n(\vec{X}) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{jn}^{\alpha_{ij}}(\vec{X})$$

where k *and* m *are constant and the bitsize of the* α_{ij}, *the sparsity of the polynomials* f_{jn} *and their constant-free arithmetic circuit complexity are all bounded by a polynomial function* $Q(n)$.

Proof. Assume by contradiction that $(\mathrm{PER}_n) \in \mathsf{mSPS}(k, m)$. By the previous remark, this implies that PER_n can be computed by polynomial size constant-free arithmetic circuits. As in the proofs of Theorem 4.1 and 1.2 in [7], it follows from this property that there is a family $(G_n(X_0, \ldots, X_n))$ in VNP such that

$$\mathrm{PW}_n(X) = G_n(X^{2^0}, X^{2^1}, \ldots, X^{2^n}). \tag{4}$$

Since the permanent is complete for VNP, we have a polynomial h such that

$$\mathrm{PER}_{h(n)}(z_1, \ldots, z_{h(n)^2}) = G_n(X_0, \ldots, X_n) \tag{5}$$

where the z_i's are either variables of G_n or constants. By hypothesis $(\mathrm{PER}_n) \in \mathsf{mSPS}(k, m)$. Let Q be the corresponding polynomial from Definition 12. From this definition and from (4) and (5) we have

$$\mathrm{PW}_n(X) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_{jn}(X)^{\alpha_{ij}}$$

where $f_{jn}(X)$ is $Q(h(n))$-sparse. This shows that the polynomial PW_n is in $\mathsf{SPS}(k, m, R(n))$ where $R(n) = Q(h(n))$.

We have proved in Theorem 10 that polynomials in $\mathsf{SPS}(k, m, R(n))$ have at most $r(n) = C \times ((m+2)R(n))^m)^{2^{k-1}-1}$ real roots. On the other hand, by construction the polynomial PW_n has 2^n roots, which is larger than $r(n)$ for all large enough n. This yields a contradiction and completes the proof of the theorem. ◄

Theorem 15 gives a lower bound for a restricted class of depth-4 circuits: The top fan-in is bounded by k, and the gates at depth 2 compute only m distinct polynomials f_j. Yet, each f_j can be duplicated an exponential number of times so that the gates at depth 3 have an unbounded fan-in. Therefore, the lower bound holds for a class of exponential-size depth-4 circuits. Note that the result is already non trivial for polynomial-size depth-4 circuits of this kind.

▶ Remark. It is possible to relax condition (iv) in Definition 12. We can replace it by the less restrictive condition:

(iv') the polynomial f_{jn} is $Q(n)$-sparse,

i.e., we allow polynomials f_{jn} with arbitrary complex coefficients. Theorem 15 still applies to this larger version of the class $\mathsf{mSPS}(k, m)$, but for the proof to go through we need to assume the Generalized Riemann Hypothesis. The only change is at the beginning of the proof: Assuming that the permanent family belongs to the (redefined) class $\mathsf{mSPS}(k, m)$, we can conclude that this family can be computed by polynomial size arithmetic circuits with arbitrary constants. To see this, note that any non-multilinear monomial in any f_{jn} can be deleted since it cannot contribute to the final result (the permanent is multilinear). And

since f_{jn} is sparse, there is a polynomial size arithmetic circuit with arbitrary constants to compute its multilinear monomials. The remainder of the proof is essentially unchanged. But to deal with arithmetic circuits with arbitrary constants (from the complex field) instead of constant-free arithmetic circuits, we shall use Corollary 4.2 of [7] instead of Theorems 1.2 and 4.1. This means that we have to assume GRH as in this corollary. It is an intriguing question whether this assumption can be removed from Corollary 4.2 of [7] and from this lower bound result.

4   Polynomial Identity Testing

This section is devoted to a proof that Identity Testing can be done in deterministic polynomial time on the polynomials studied in the previous sections. Recall from Definition 5 that for $\phi = \sum_{i=1}^{k} P_i \in \mathsf{SPS}(k, m, t)$, (ϕ_n) is defined by $\phi_n = \sum_{i=n}^{k} g_i^{(n)} P_i$.

▶ **Lemma 16.** *Let $\phi \in \mathsf{SPS}(k, m, t)$ and (ϕ_n) as in Definition 5. Then for $l < k$, $\phi_l \equiv 0$ if and only if $\phi_{l+1} \equiv 0$ and ϕ_l has a smaller degree than $g_l^{(l)} P_l$.*

Proof. If for all i, $g_i^{(l)}$ is identically zero, then the lemma holds. If there is at least one which is not identically zero, assume that it is $g_l^{(l)}$ up to a reindexing of the terms.

Let $T_l = g_l^{(l)} P_l$, recall that $\phi_{l+1} = g_l T_l \pi (\phi_l / T_l)'$. If $\phi_l \equiv 0$, then $\phi_{l+1} \equiv 0$. Moreover, we have assumed that $T_l \not\equiv 0$ and it is thus of larger degree than ϕ_l which is identically 0.

Assume now that $\phi_{l+1} \equiv 0$, that is $g_l T_l \pi (\phi_l / T_l)' \equiv 0$. By hypothesis, T_l and π are not identically zero, therefore $(\phi_l / T_l)' \equiv 0$. Thus there is $\lambda \in \mathbb{R}$ such that $\phi_l = \lambda T_l$. Since by hypothesis ϕ_l and T_l have different degrees, $\lambda = 0$ and $\phi_l \equiv 0$.   ◀

To solve PIT, we will need to explicitly compute the sequence of polynomials ϕ_l. Thus, the algorithm is not black-box: it must have access to a representation of the input polynomial under form (1).

▶ **Theorem 17.** *Let k and m be two integers and $\phi \in \mathsf{SPS}(k, m, t)$: we have $\phi = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}}$ where for all i and j, f_j is t-sparse and $\alpha_{ij} \geq 0$. Then one can test if ϕ is identically zero in time polynomial in t, in the size of the sparse representation of the f_j's and in the α_{ij}'s.*

Proof. Let (ϕ_n) be the sequence defined from ϕ as in Definition 5. Lemma 16 implies that ϕ is identically zero if and only if ϕ_k is identically zero and that for all $l < k$, $\phi_l = \sum_{i=l}^{k} g_i^{(l)} P_i$ has a strictly smaller degree than $g_l^{(l)} P_l$.

First, one computes the sparse polynomials $g_i^{(l)}$ for all i and l as sums of monomials. It is done in time polynomial in the size of the f_j's since k and m are fixed. We can then verify if ϕ_k is identically zero.

We now want to test for each l if the degrees of $g_l^{(l)} P_l$ and of ϕ_l differ. First remark that we know the highest degree monomial of $g_i^{(l)}$ for $i \geq l$ since we have computed all the $g_i^{(l)}$'s. One can also compute the highest degree monomial of each P_i in time polynomial in the α_{ij}'s (not their bitsize) and the size of the f_j's. We have thus computed the degree of $g_i^{(l)} P_i$ for all i and l and we reorder them so that $g_l^{(l)} P_l$ is of highest degree amongst them. Let S denote the sum of the highest degree monomial of $g_i^{(l)} P_i$ for $i \geq l$ that we have computed. Since the degree of $g_l^{(l)} P_l$ is maximum, we have $\deg(\phi_l) < \deg(g_l^{(l)} P_l)$ if and only if $\deg(S) < \deg(g_l^{(l)} P_l)$ and we can test the latter condition since we have computed these polynomials explicitly.   ◀

This algorithm is polynomial in the α_{ij}'s, though ideally we would like it to be polynomial in their bitsize.

▶ **Proposition 18.** *Assume that we have access to an oracle which decides whether*

$$\sum_{i=1}^{k} \prod_{j=1}^{m} a_{ij}^{\alpha_{ij}} = 0. \tag{6}$$

Let $\phi = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}}$ as in Theorem 17. Then one can decide deterministically whether ϕ is identically zero in time polynomial in the sparsity of the f_j's and in the bitsize of the a_{ij}'s and α_{ij}'s.

Proof. The only dependency in the α_{ij}'s in the proof of Theorem 17 is the computation of the coefficient of the highest degree monomials of the $g_i^{(l)} P_i$. With the oracle for (6), we skip this step and achieve a polynomial dependency in the bitsize of the α_{ij}'s.   ◀

A direct computation of the constant on the left-hand side of (6) is not possible since it involves numbers of exponential bitsize (the exponents α_{ij} are given in binary notation). The test to 0 can be made by computing modulo random primes, but this is ruled out since we want a deterministic algorithm. Note also that this test is a PIT problem for polynomials in $\mathsf{SPS}(k, m, t)$ where the f_j's are constant polynomials. For general arithmetic circuits, it is likewise known that PIT reduces to the case of circuits without any variable occurrence ([4], Proposition 2.2).

The polynomial identity test from Theorem 17 can also be applied to the class of multivariate polynomial families $\mathsf{mSPS}(k, m)$ introduced in the previous section. Indeed, let $P(X_1, \ldots, X_n) = \sum_i \prod_j f_j^{\alpha_{ij}}$ belongs to some $\mathsf{mSPS}(k, m)$ family, and suppose we know a bound d on its degree. We turn P into a univariate polynomial Q by the classical substitution (sometimes attributed to Kronecker) $X_i \mapsto X^{(d+1)^i}$. We write $Q(X) = \sum_i \prod_j g_j^{\alpha_{ij}}$, where each univariate polynomial g_j is the image of f_j by the substitution. It is a folklore result that $P \equiv 0$ if and only if $Q \equiv 0$, thus we can apply the PIT algorithm of Theorem 17 on Q.

Let s be the size of the representation of P, meaning that P depends on at most s variables, the f_j's have a constant-free circuit of size at most s and are s-sparse, and the α_{ij} are at most equal to s. (Note that we do not bound their bitsizes but their values as it is needed for our PIT algorithm.) Then the degree of the f_j's is at most 2^s, and $d \leq 2^{\mathsf{poly}(s)}$ where $\mathsf{poly}(s)$ denotes some polynomial function of s. The g_j's therefore have a degree at most $2^{s\mathsf{poly}(s)} \times 2^s = 2^{s\mathsf{poly}(s)+s}$. This proves that Q satisfies the hypothesis of Theorem 17.

5   Conclusion

We have shown that the real τ-conjecture from [14] holds true for a restricted class of polynomials, and from this result we have obtained an identity testing algorithm and a lower bound for the permanent. Other simple cases of the conjecture remain open. In the general case, we can expand a sum of product of sparse polynomials as a sum of at most kt^m monomials. There are therefore at most $2kt^m - 1$ real roots. As pointed out in [14], the case $k = 2$ is already open: is there a polynomial bound on the number of real roots in this case? Even simpler versions of this question are open. For instance, we can ask whether the number of real roots of an expression of the form $f_1 \cdots f_m + 1$ is polynomial in m and t. A bare bones version of this problem was pointed out by Arkadev Chattopadhyay (personal communication): taking $m = 2$, we can ask what is the maximum number of real roots of an expression of the form $f_1 f_2 + 1$. Expansion as a sum of monomials yields a $O(t^2)$ upper bound, but for all we know the true bound could be $O(t)$.

―― **References** ―――――――――――――――――――――――――――――

1 M. Agrawal. Proving lower bounds via pseudo-random generators. In *Proceedings of the 25th conference of the Foundations of Software Technology and Theoretical Computer Science*, pages 92–105. Springer, 2005.

2 M. Agrawal and R. Saptharishi. Classifying Polynomials and Identity Testing. In *Current Trends in Science*, pages 149–162. Indian Academy of Sciences, 2009.

3 M. Agrawal and V. Vinay. Arithmetic circuits: A chasm at depth four. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 67–75, 2008.

4 E. Allender, P. Bürgisser, J. Kjeldgaard-Pedersen, and P. Bro-Miltersen. On the complexity of numerical analysis. *SIAM Journal on Computing*, 38(5):1987–2006, 2009. Conference version in CCC 2006.

5 M. Beecken, J. Mittmann, and N. Saxena. Algebraic independence and blackbox identity testing. *Proceedings of the 38th International Colloquium on Automata, Languages and Programming*, 2011. Arxiv preprint arXiv:1102.2789.

6 A. Borodin and S. Cook. On the number additions to compute specific polynomials. *SIAM Journal on Computing*, 5(1):146–157, 1976.

7 P. Bürgisser. On defining integers and proving arithmetic circuit lower bounds. *Computational Complexity*, 18(1):81–103, 2009.

8 P. Bürgisser, M. Clausen, and M. A. Shokrollahi. *Algebraic Complexity Theory*. Springer, 1997.

9 P. Bürgisser. *Completeness and Reduction in Algebraic Complexity Theory*. Algorithms and Computation in Mathematics. Springer, 2000.

10 B. Grenet, P. Koiran, N. Portier, and Y. Strozecki. The Limited Power of Powering: Polynomial Identity Testing and a Depth-four Lower Bound for the Permanent. Manuscript, 2011. http://arxiv.org/abs/1107.1434.

11 J. Heintz and C.-P. Schnorr. Testing polynomials which are easy to compute. In *Logic and Algorithmic (an International Symposium held in honour of Ernst Specker)*, pages 237–254. Monographie n° 30 de L'Enseignement Mathématique, 1982. Preliminary version in *Proc. 12th ACM Symposium on Theory of Computing*, pages 262-272, 1980.

12 V. Kabanets and R. Impagliazzo. Derandomizing polynomial identity tests means proving circuit lower bounds. *Computational Complexity*, 13(1):1–46, 2004.

13 P. Koiran. Arithmetic circuits: the chasm at depth four gets wider. *Arxiv preprint arXiv:1006.4700*, 2010.

14 P. Koiran. Shallow circuits with high-powered inputs. *Proceedings of the Second Symposium on Innovations in Computer Science*, 2011.

15 T.Y. Li, J.M. Rojas, and X. Wang. Counting real connected components of trinomial curve intersections and m-nomial hypersurfaces. *Discrete and computational geometry*, 30(3):379–414, 2003.

16 N. Saxena. Progress on Polynomial Identity Testing. *Bull. EATCS*, 99:49–79, 2009.

17 J. T. Schwartz. Fast probabilistic algorithms for verification of polynomials identities. *Journal of the ACM*, 27:701–717, 1980.

18 M. Shub and S. Smale. On the intractability of Hilbert's Nullstellensatz and an algebraic version of "P=NP". *Duke Mathematical Journal*, 81(1):47–54, 1995.

19 S. Smale. Mathematical problems for the next century. *The Mathematical Intelligencer*, 20(2):7–15, 1998.

20 L.G. Valiant. Completeness classes in algebra. In *Proceedings of the 11th Annual ACM Symposium on Theory of Computing*, pages 249–261, 1979.