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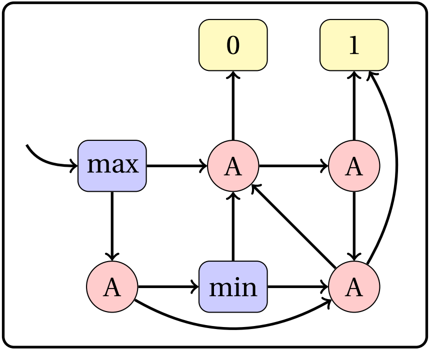
Almost Acyclic Simple Stochastic Games

Mardi 17 Décembre,
Journées CMF

Simple stochastic game (SSG)

A Simple Stochastic Game (Shapley, Condon) is defined by a directed graph with :

- three sets of vertices V_{MAX}, V_{MIN}, V_{AVE} of outdegree 2
- two (or more) 'sink' vertices with rational values



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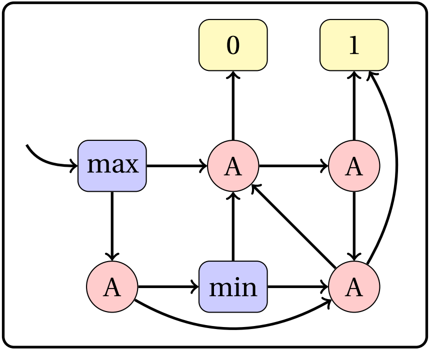
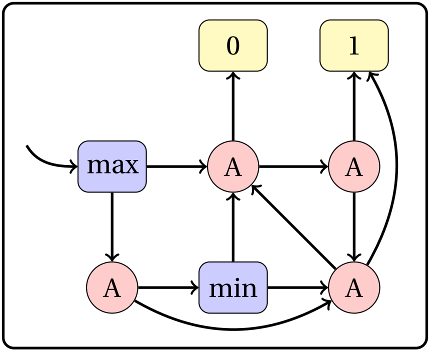


Two players : MAX and MIN, and randomness.

Rules of a SSG

A play consists in moving a pebble on the graph :

- player MAX wants to maximize the value of the sink reached.
- player MIN wants to minimize the value. If no sink is reached, the value is 0.



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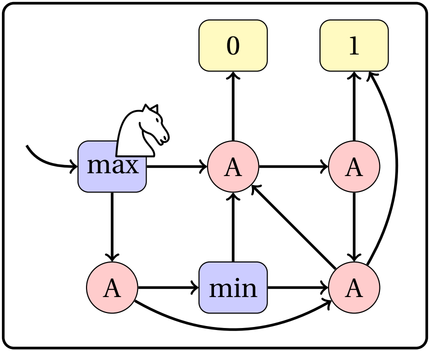

On a MAX node player MAX decides where to go next.

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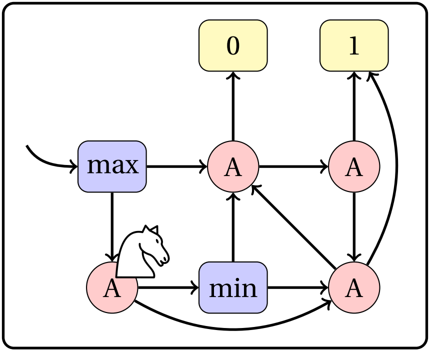

On a AVE node the next vertex is randomly determined.

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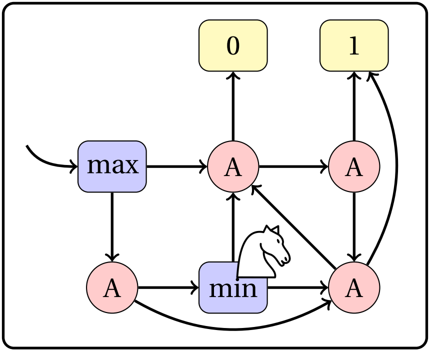

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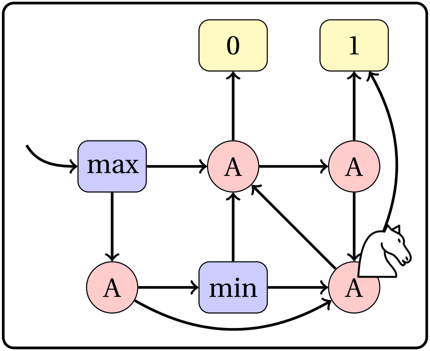

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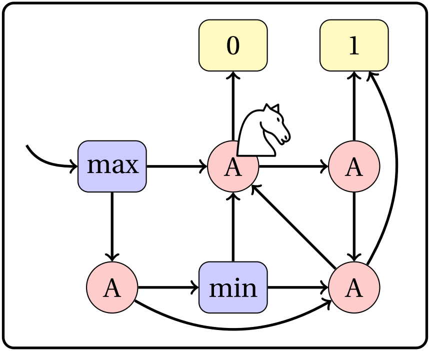

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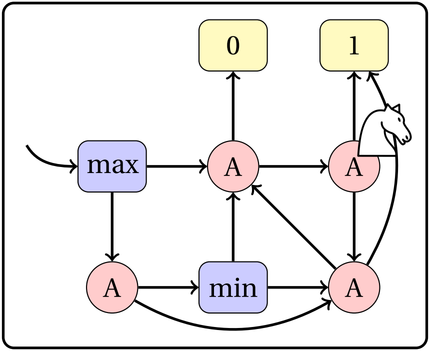

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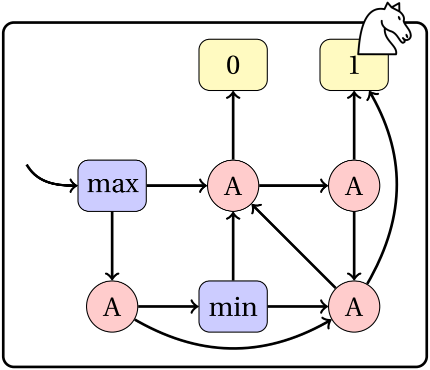

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Strategies and values

General definition of a **strategy** σ for a player MAX :

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The **value** of a vertex x is the best expected value of a sink that MAX can guarantee starting from x :

$$v(x) = \sup_{\substack{\sigma \text{ strategy} \\ \text{for MAX}}} \inf_{\substack{\tau \text{ strategy} \\ \text{for MIN}}} \underbrace{\mathbb{E}_{\sigma,\tau}\left(\text{value of the sink reached} \mid \text{game starts in } x\right)}_{v_{\sigma,\tau}(x)}$$

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Decision problem : $v(x) > 0.5$?

Why simple stochastic games ?

They generalize important models such as :

- Parity games (model checking of μ-calculus)
- Mean payoff games (useful for optimisation)
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Also there are :

- An example of a problem yet between P and NP
- A simple framework to study stochastic games
- A good model to study partial information

Simpler game : Stopping SSGs

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Theorem (Condon 89)

For every SSG G, there is a polynomial-time computable SSG G' such that

- *G' is stopping*
- *size of G' = poly(size of G)*
- *for all vertices x,* $v_{G'}(x) > \frac{1}{2}$ *if and only if* $v_G(x) > \frac{1}{2}$

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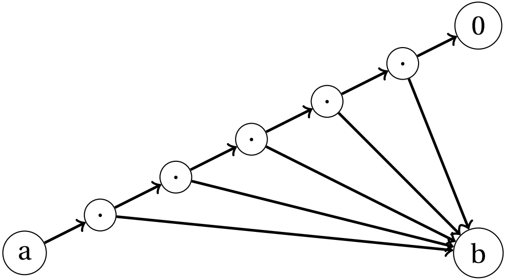

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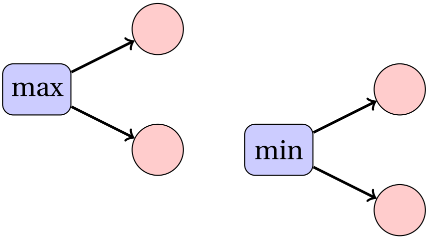
We call them **positional strategies** for short.

$$\sigma : V_{\mathrm{MAX}} \longrightarrow V, \quad \tau : V_{\mathrm{MIN}} \longrightarrow V$$



Optimality of positional strategies

Lemma

Against a **positional** *strategy* σ, MIN *might as well respond positional :*

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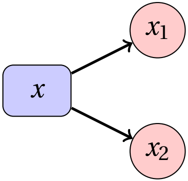
Bellman equation characterizes optimality :

$$v^* = \min_{\tau \text{ general}} v_{\sigma,\tau} \Leftrightarrow \forall x, v^*(x) = \begin{cases} \min(v^*(x_1), v^*(x_2)) & \text{if } x \in \text{MIN} \\ \frac{1}{2}(v^*(x_1) + v^*(x_2)) & \text{if } x \in \text{AVE} \\ 0 \text{ or } 1 & \text{if } x \in \text{SINK} \end{cases}$$



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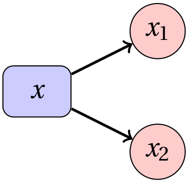

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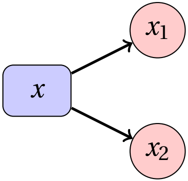
An optimal positional strategy consists in playing optimally at each node wrt to v^*.



Minimax Theorem

Theorem (Condon 89)

For all vertices x,

$$
\begin{aligned}
v(x) &= \sup_{\sigma \text{ general}} \inf_{\tau \text{ general}} v_{\sigma,\tau}(x) \\
&= \inf_{\tau \text{ general}} \sup_{\sigma \text{ general}} v_{\sigma,\tau}(x) \\
&= \max_{\sigma \text{ positional}} \min_{\tau \text{ positional}} v_{\sigma,\tau}(x) \\
&= \min_{\tau \text{ positional}} \max_{\sigma \text{ positional}} v_{\sigma,\tau}(x)
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Complexity Considerations

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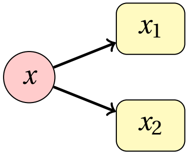
Open question : is the value problem in P ?

Computing values

Fix σ, τ positional strategies.

- if $x \in \text{MAX}$, $v_{\sigma,\tau}(x) = v_{\sigma,\tau}(\sigma(x))$
- if $x \in \text{MIN}$, $v_{\sigma,\tau}(x) = v_{\sigma,\tau}(\tau(x))$
- if $x \in \text{AVE}$, $v_{\sigma,\tau}(x) = \frac{1}{2} v_{\sigma,\tau}(x_1) + \frac{1}{2} v_{\sigma,\tau}(x_2)$
- if $x \in \text{SINK}$, $v_{\sigma,\tau}(x) \in [0, 1]$

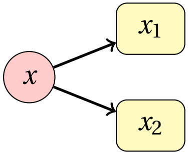


This amounts to solve a linear system.

Computing values

Fix σ only (best response).

- if $x \in \mathrm{MAX}$, $v_\sigma(x) = v_\sigma(\sigma(x))$
- if $x \in \mathrm{MIN}$, $v_\sigma(x) \leq v_\sigma(x_1)$ and $v_\sigma(x) \leq v_\sigma(x_2)$
- if $x \in \mathrm{AVE}$, $v_\sigma(x) = \frac{1}{2} v_{\sigma,}(x_1) + \frac{1}{2} v_\sigma(x_2)$
- if $x \in \mathrm{SINK}$, $v_\sigma(x) \in [0,1]$
- $\max \sum_{x \in \mathrm{MIN}} v_\sigma(x)$

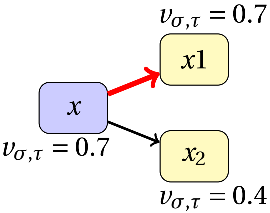


This amounts to solve a linear program.

The switch operation

x is a MIN vertex and $v_{\sigma,\tau}(x) = v_{\sigma,\tau}(x_1) > v_{\sigma,\tau}(x_2)$

switching τ at $x : \tau'(x) = x_2$ and equal to τ elsewhere.



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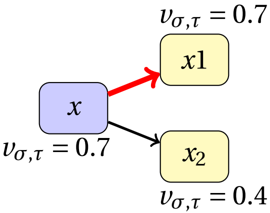

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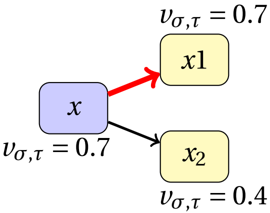

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Optimality condition : no switch .
Algorithm to find an optimal strategy against σ : keep switching.

Strategy improvement algorithms

The strategy improvement algorithm a.k.a Hoffman-Karp algorithm (1966, MDP context) is

1. choose τ_0 and let $\sigma_0 = \sigma(\tau_0)$ (best response)
2. while (σ_k, τ_k) is not optimal, obtain τ_{k+1} by switching τ_k; let $\sigma_{k+1} = \sigma(\tau_{k+1})$

based on :

Lemma

$v_{\sigma_{k+1},\tau_{k+1}} < v_{\sigma_k,\tau_k}$ as long as (σ_k, τ_k) is not optimal.

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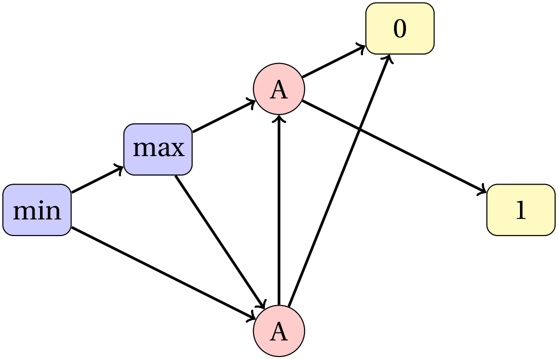
When the algorithm ends, say at (σ^*, τ^*), each one plays optimally :

$$v_{\sigma^*,\tau^*} = \max_{\sigma \text{ pos}} \min_{\tau \text{ pos}} v_{\sigma,\tau} = \min_{\tau \text{ pos}} \max_{\sigma \text{ pos}} v_{\sigma,\tau}$$

Solving an acyclic SSG in linear time



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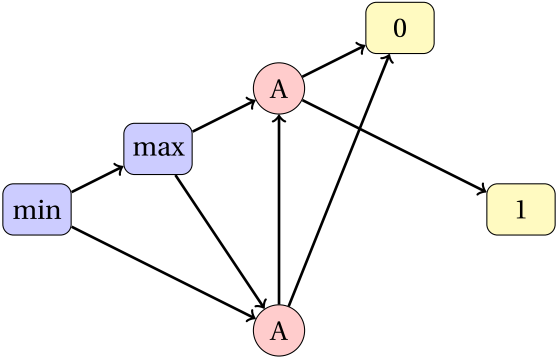

No cycle : compute the values backward from the sinks in time $O(n)$.

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Need to compute values $\Rightarrow O(n^4|V_{MAX}|)$.

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algo for checking **A2** :

1. solve the acyclic SSG obtained when an arbitrary MAX vertex is fixed to open
2. if **A2** is true, the next open MAX vertex (say x) is also open in the 1-cycle SSG
3. so check it by solving the acyclic SSG forced to be open at x.

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Based on the previous algo, by opening a vertex next to each fork vertex :

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To be compared to the strategy improvment algorithm : $O(n^4 2^k)$

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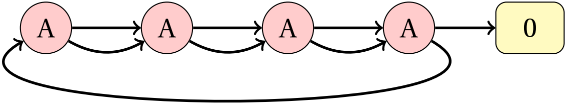

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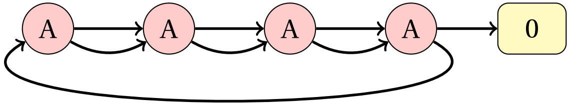

Bisection algo with $k = 1$: x is the vertex to remove to get an acyclic SSG

1. solve the acyclic SSG obtained when x is replaced by a sink with value $s = \frac{min+max}{2}$

SSG with feedback vertex set of size k

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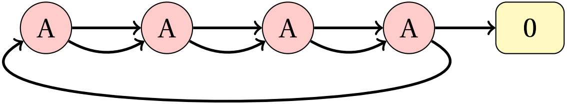

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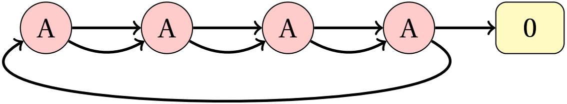

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3. otherwise if $s > \min(v(x_1), v(x_2)) + \epsilon \Rightarrow$ then the value of x is less than s, go back to 1 with $max = s$.

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The method can be used to remove k vertices in any SSG and thus makes other classes of SSG tractable.

Tanks for listening !

Questions ?