Enumerating models of a DNF: sublinear algorithms

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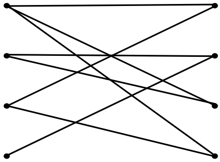
Séminaire LIMOS

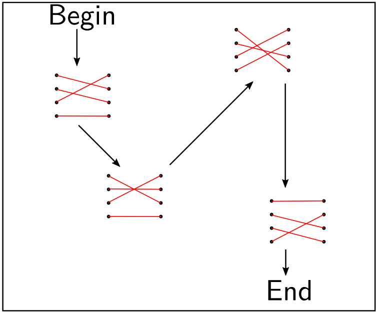
Enumeration problems

- **Enumeration problems**: list all solutions rather than deciding whether there is one or finding one.
- Motivations: database queries, counting, optimization, building libraries, datamining.
- Complexity measures: total time and delay between solutions.

Perfect matchings?



Solution space:



A concrete example: enumerating integers

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Complexity: The delay, going from a number to the next, is in $O(n)$ in the worst case:

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However, the average time per integer generated is constant:

$$\sum i2^{-i} \leq 2$$

Faster enumeration using Gray code

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There is a constant time algorithm to find the next element in the Gray code:

- the number of one is even: flip the last bit
- the number of one is odd: flip the bit to the left of the rightmost 1

Framework

An enumeration problem A is a function which associates to each input x a set of solutions $A(x)$.

An enumeration algorithm must generate every element of $A(x)$ one after the other without repetition.

The computation model for enumeration is a RAM with uniform cost measure and an OUPTPUT instruction. Support efficient data structures.

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Complexity measures:

- total time
- delay
- space

Parameters:

- input size
- output size
- single solution size

Generating unions

Closure by union: given a collection of sets, generate all unions of these sets.
Instance: a set $S = \{s_1, \ldots s_m\}$ with $s_i \subseteq \{1, \ldots, n\}$.
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Problem: list all distinct unions of elements in S.

Use a saturation algorithm:

- **begin** with a polynomial number of simple solutions: the sets s_i
- **for each** k-uple of already generated solutions apply a rule to produce a new solution: produce $s \cup s'$ for each pair (s, s') of solutions
- **stop** when no new solutions are found

Unions in polynomial delay

A better algorithm to compute the closure by union.

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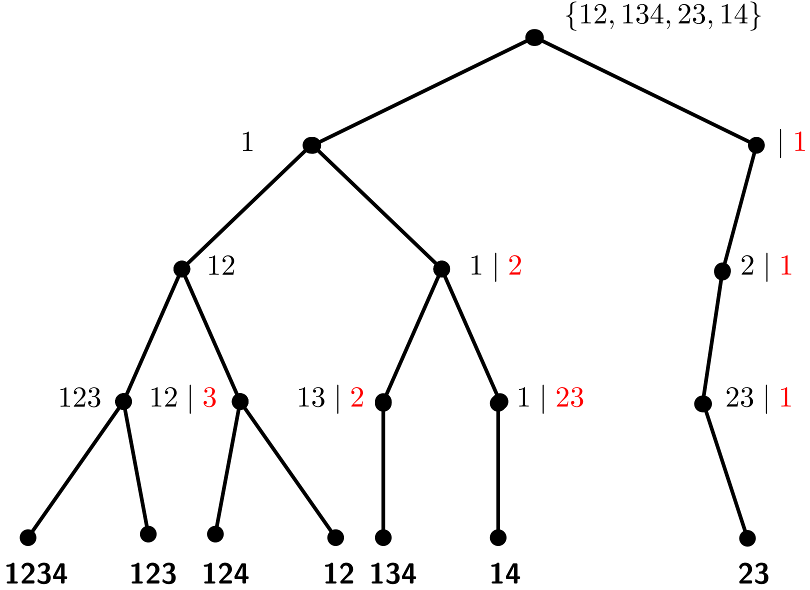
4. Easy to solve in $O(mn)$: compute

$$A' = \bigcup_{s \in S, s \cap B = \emptyset} s$$

   Delay equal *depth of the recursive tree* time *cost of solving extension*: $O(mn^2)$.

Partial solution tree



Polynomial delay methods

Flashlight search can be improved by:

- Reducing the complexity of the extension problem or the depth of the tree
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Other methods:

1. Solutions organized in a tree (models of a $2 - CNF$).
2. Solutions organized in a connected graph, with small or enumerable neighboroods. Reverse search (maximal cliques).

What is a really efficient enumeration algorithm ?

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A typical example: listing all paths in a DAG by a DFS.

Enumerating the models of a DNF

- A term is a conjunction of literals over n variables.
- A DNF formula is a disjunction of m terms.
- $\textsc{Enum}{\cdot}DNF$ is the problem of enumerating satisfying assignments ($=$ models) of a DNF.

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Why is this problem interesting?

- Extremely simple: solution of terms in constant delay. Union of regular sets of solutions while dealing with repetitions.
- DNF enumeration is connected to knowledge representation, minimal transversal enumeration, subset membership queries, CQ + SO variables, DNF model counting, PAC-learning . . .

Representation of a DNF

A DNF formula D is characterized by the following parameters:

1. n the number of variables
2. m the number of terms
3. $||D||$ the sum of the sizes of the terms, $||D|| \leq nm$
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Data structure for DNF: the Trie

A trie or prefix tree is a tree labeled by letters. It represents the set of words associated to its paths from the root.

A DNF formula or a set of models can be represented by a trie. It supports insertion, lookup and deletion in time $O(n)$.

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Simplest algorithm: for each term, generate all its models and deal with repetitions using a set data structure.

The cost for lookup and insertion depends on the data structure:

- Array (sorted or not): $O(ns)$
- Binary search tree: $O(n \log(s))$
- Hash table: expected $O(n)$
- Trie: $O(n)$

Flashlight for DNF

Application of the flashlight method: depth of the tree n, extension problem in $O(mn)$: delay in $O(mn^2)$.

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Better data structure (similar to the one for monotone CNF [Uno]):

- A counter for the number of satisfiable terms in D: c_D
- For each term T a counter of falsified variables: c_T
- For each literal, the list of terms where it appears

In a branch, each literal of a term is visited once at most: delay $O(||D||)$.

Lower Bound Conjectures

We have a linear delay algorithm. Can we get rid of m in the delay?

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DNF Enumeration Conjecture

$\textsc{Enum}{\cdot}DNF \notin \textsc{SDelayP}$.

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There is no algorithm solving $\textsc{Enum}{\cdot}DNF$ in delay $o(m)p(n)$ where m is the number of terms, n the number of variables and p a polynomial.

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Stronger conjectures can be made by restricting to subclasses of DNF and to average delay. We refute some of them in this presentation.

DNF with small terms

Definition

A term T is a k-term if $|T| \leq k$.
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Solutions aplenty: A k-term has 2^{n-k} models.

A compact representation of subproblems: if D is represented by the trie $T(D)$, one can compute $T(D[x \to 0])$ and $T(D[x \to 1])$ in time $O(||D||)$.

All removed terms are stored to be able to insert them back later.

Branching along a term

Let $T = x_1 \wedge x_2 \cdots \wedge x_k$ be a term of D. The models of D can be partitionned into $k+1$ disjoint subsets, models of:

- $D \wedge \bar{x}_1$
- $D \wedge x_1 \wedge \bar{x}_2$
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- $D \wedge x_1 \wedge x_2 \cdots \wedge x_k$

The models of $D \wedge x_1 \wedge x_2 \cdots \wedge x_k$ are the models of $x_1 \wedge x_2 \cdots \wedge x_k$ an can be enumerated in constant delay.

Enumeration for k-DNF

Theorem

The models of a k-DNF with n variables can be enumerated with precomputation in $O(n)$ and $O(k^{3/2}2^{2k})$ delay.

Sketch of the algorithm:

- Flashligth algorithm, *branch along a term* of the current formula.
- When a variable is fixed, *compute the trie of the reduced formula*.
- *Interleave* enumeration of models of a term T and branching along T.
- Balance the cost of maintaining a trie and the number of solutions given by a model, worst case $n = 2k$.

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Lemma

Let $\gamma = \log_3(2)$. A DNF formula with m non empty distinct terms has at least m^γ models.

Proof sketch:
Induction on the number of variables. Cut the formula in three parts: terms with x_1, $\bar{x}_1$, whithout x_1 or $\bar{x}_1$. Evaluate how the three subformulas contribute models to the original formula and apply inequalities.

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Same idea as for k-DNF, amortize the cost of branching.

Theorem

The models of a DNF can be enumerated with average delay $O(n^2 m^{1-\gamma})$ and polynomial space.

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Proof sketch:

Flashlight search, reduction of the trie when fixing a variable.
Branching cost $O(mn)$, but can be amortized over m^γ models.

Improving the average delay

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Differentiate between **fast branching** (the size of one subformula decreases by a factor $1/2$) and **slow branching**. A different way of counting the complexity of branching and amortizing it is used in the two cases.

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Above algorithm and the one for k-DNF can be combined.

Theorem

There is an algorithm with average delay $O(2^{3k/2})$ to enumerate the models of a k-DNF.

Monotone DNF

Problems of generating ideals in a boolean lattice given by an antichain.

- Remove redundant terms (terms included in other) in time $O(m^2)$.
- Each term has one proper model (the minimal one).
- For each term, enumerate its models in lexicographic order and store them in a set.
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Open question: Can the preprocessing be improved? Can the delay be improved? Can the space be made polynomial?

Average delay for Monotone DNF

The algorithm for general DNF is adapted using two ideas.

- Each term has a minimal model not shared with the other terms. For an instance with m terms, at least m models.
- Terms of small size have many solutions: a formula with a small term cost "nothing" in the flashlight. If all terms are large represent them by their complementary.

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Theorem

There is an algorithm to enumerate the models of a monotone DNF with polynomial space and average delay $O(mn)$.

Results [Capelli, S. 2019]

Class	**Delay**	**Space**
DNF	$O(\|\|D\|\|)$	$O(\|\|D\|\|)$
$(\star)$ DNF	$O(nm^{1-\gamma})$ average delay	$O(\|\|D\|\|)$
$(\star)$ k-DNF	$k^{3/2}2^{2k}$	$O(\|\|D\|\|)$
$(\star)$ Monotone DNF	$O(n^2)$, m^2 preprocessing	$O(sn)$
$(\star)$ Monotone DNF	$O(\log(mn))$ average delay	$O(mn)$

Table: Overview of the results. In this table, D is a DNF, n its number of variables and m its number of terms. New contributions are annotated with $(\star)$.

Thanks!
Questions?